

Question 1(a)(i)

$$\frac{d}{dx} \ln(\ln x) = \frac{1}{\ln x} \left(\frac{d}{dx} \ln x \right) = \frac{1}{x \ln x}.$$

Question 1(a)(ii)

$$\frac{dy}{dx} + \frac{1}{x \ln x} y = \frac{x}{\ln x}.$$

Integrating factor = $e^{\int \frac{1}{x \ln x}} = e^{\ln \ln x} = \ln x$. Hence,

$$\begin{aligned} y \ln x &= \int \frac{x}{\ln x} \ln x = \frac{x^2}{2} + C \\ \implies y(e) \ln e = 0 &= \frac{e^2}{2} + C \implies C = -\frac{e^2}{2}. \end{aligned}$$

$$\begin{aligned} y \ln x &= \frac{x^2}{2} - \frac{e^2}{2} \\ \implies y(e^2) \ln e^2 &= \frac{e^4}{2} - \frac{e^2}{2} \\ \implies y(e^2) &= \frac{e^4 - e^2}{4}. \end{aligned}$$

Question 1(b)

$$\begin{aligned}\frac{dN}{N(N-100)} = 0.01dt &\implies \left(\frac{1}{N-100} - \frac{1}{N} \right) dN = dt \\ &\implies \ln |N-100| - \ln |N| = t + C \\ &\implies \frac{N-100}{N} = Ae^t.\end{aligned}$$

$$N(0) = 101 \implies A = 1/101. \text{ Therefore,}$$

$$\begin{aligned}\frac{N-100}{N} = \frac{1}{101}e^t &\implies N = \frac{10100}{101 - e^t} \\ &\implies N(4.61) = \frac{10100}{101 - e^{4.61}} \approx 19579.\end{aligned}$$

Question 2(a)

For homogeneous solution y_h , solve

$$\lambda^2 + 2\lambda + 1 = 0 \implies \lambda = -1.$$

Hence $y_h = (Ax + B)e^{-x}$. For y_p , try

$$\begin{aligned} y = ue^{-x} &\implies y' = u'e^{-x} - ue^{-x} \\ &\implies y'' = u''e^{-x} - 2u'e^{-x} + ue^{-x}. \end{aligned}$$

So

$$\begin{aligned} u''e^{-x} - 2u'e^{-x} + ue^{-x} + 2(u'e^{-x} - ue^{-x}) + ue^{-x} &= xe^{-x} \\ \implies u'' = x &\implies u = \frac{1}{6}x^3 + Cx + D. \end{aligned}$$

Thus $y = (Ax + B)e^{-x} + \frac{1}{6}x^3e^{-x}$.

$$y(0) = 1 \implies B = 1.$$

Since

$$y' = Ae^{-x} - (Ax + 1)e^{-x} + \frac{1}{2}x^2e^{-x} - \frac{1}{6}x^3e^{-x}.$$

$$y'(0) = 0 \implies A - 1 = 0.$$

Finally,

$$y = (x + 1)e^{-x} + \frac{1}{6}x^3e^{-x} = (1 + x + \frac{1}{6}x^3)e^{-x}.$$

Question 2(b)

For homogeneous solution y_h , solve

$$\lambda^2 + 1 = 0 \implies \lambda = \pm i.$$

Hence $y_h = A \cos x + B \sin x$.

Wronskian $W = \cos^2 x - (-\sin^2 x) = 1$, so

$$\begin{aligned} u &= - \int \sin x \sec x = - \int \frac{\sin x}{\cos x} = \ln(\cos x) \\ v &= \int \cos x \sec x = \int 1 = x. \end{aligned}$$

Hence $y = A \cos x + B \sin x + \ln(\cos x) \cos x + x \sin x$.

$$y(0) = 1 \implies A = 1.$$

$$y' = -\sin x + B \cos x - \ln(\cos x) \sin x - \tan x \cos x + \sin x + x \cos x.$$

$$y'(0) = 1 \implies 1 = B. \text{ Hence}$$

$$y = \cos x + \sin x + \ln(\cos x) \cos x + x \sin x.$$

Question 3(a)(i)

$$0 < x < \frac{\pi}{2} \implies 0 < \sin x < 1 \implies 0 < \pi \sin x < \pi.$$

$$\text{hence } \cos(\pi \sin x) = 0 \implies \pi \sin x = \frac{\pi}{2} \implies x = \frac{\pi}{6}.$$

Using Taylor's theorem,

$$\begin{aligned} \ddot{x} &\approx \left. \frac{d}{dx} \cos(\pi \sin x) \right|_{x=\frac{\pi}{6}} \left(x - \frac{\pi}{6} \right) \\ &= -\sin(\pi \sin x) \pi \cos x \Big|_{x=\frac{\pi}{6}} \left(x - \frac{\pi}{6} \right) \\ &= -\sin\left(\pi\left(\frac{1}{2}\right)\right) \pi \cos\left(\frac{\pi}{6}\right) \left(x - \frac{\pi}{6} \right) \\ &= -\frac{\sqrt{3}}{2} \pi \left(x - \frac{\pi}{6} \right). \end{aligned}$$

Set

$$y = x - \frac{\pi}{6} \implies \ddot{y} \approx -\frac{\sqrt{3}}{2} \pi y.$$

Hence $x = \frac{\pi}{6}$ is a stable equilibrium.

Question 3(a)(ii)

$$\omega \approx \sqrt{\frac{\sqrt{3}}{2} \pi} \approx 1.65.$$

Question 3(b)

Before the pirates,

$$\begin{aligned} N &= \hat{N} e^{(B-D)t} \\ 2\hat{N} &= \hat{N} e^{(B-D)15} \implies B - D = \frac{1}{15} \ln 2. \end{aligned}$$

After the pirates,

$$\begin{aligned} N &= \tilde{N} e^{(\tilde{B}-\tilde{D})t} \\ \frac{1}{2}\tilde{N} &= \tilde{N} e^{(\tilde{B}-\tilde{D})10} \implies \tilde{B} - \tilde{D} = -\frac{1}{10} \ln 2 \\ &\implies 0.05B - 1.5D = -\frac{1}{10} \ln 2. \end{aligned}$$

Solving for B ,

$$1.5B - 0.05B = 2\left(\frac{1}{10} \ln 2\right) \implies B \approx 0.0956.$$

or $B \approx 9.56\%$.

Question 4(a)

$$B = 0.1, N_{\infty} = 200,000 \implies s = 0.1/200000 = 5 \times 10^{-7}.$$

Hence,

$$\begin{aligned} \frac{dN}{dt} &= BN - sN^2 - E \\ &= 0.1N - 5 \times 10^{-7}N^2 - 3000. \end{aligned}$$

Since $B^2/4s = BN_{\infty}/4 = 5000 > E$, we have two equilibriums given by the roots of $BN - sN^2 - E$. Solving, this quadratic gives us two roots, $\beta_1 \approx 36754$ (unstable) and $\beta_2 \approx 163246$ (stable).

Limiting population is then ≈ 163246 .

Question 4(b)

$$\begin{aligned}\frac{dN}{dt} &= 0.001N(100 - (N - 100)^2) \\ &= -0.001N(N - 90)(N - 110).\end{aligned}$$

The curve $F(N) = \frac{dN}{dt}$ has 3 roots, giving us 3 equilibriums. $N = 90$ is an unstable equilibrium while $N = 110$ is stable.

(i) $A = 90$

(ii) If $\hat{N} > 110$ then $N \rightarrow 110$.

Question 5(a)

Apply Laplace transform to get,

$$\begin{aligned}L(y'') &= s^2Y - sy(0) - y'(0) = s^2Y - s \\L(y) &= Y \\L(0.5\delta(t - 2\pi)) &= 0.5e^{-2\pi s}.\end{aligned}$$

Hence

$$(s^2 + 1)Y = s + 0.5e^{-2\pi s} \implies Y = \frac{s}{s^2 + 1} + \frac{0.5e^{-2\pi s}}{s^2 + 1}.$$

Since $L(f(t - a)u(t - a)) = e^{-as}F(s)$ and $L(\sin t) = \frac{1}{s^2 + 1}$,

$$y = L^{-1}(Y) = \cos t + \frac{1}{2} \sin(t - 2\pi)u(t - 2\pi).$$

Question 5(b)

Completing the square,

$$\frac{e^{-\pi s}}{s^2 + 2s + 5} = \frac{e^{-\pi s}}{(s + 1)^2 + 2^2}.$$

Using s -shifting,

$$L(e^{-t} \sin 2t) = \frac{2}{(s + 1)^2 + 2^2}.$$

Using t -shifting, $L(f(t - a)u(t - a)) = e^{-as}F(s)$, we have

$$\begin{aligned} L^{-1} \left(\frac{e^{-\pi s}}{s^2 + 2s + 5} \right) &= \frac{1}{2} e^{-(t-\pi)} \sin(2(t - \pi)) u(t - \pi) \\ L^{-1} \left(\frac{e^{-\pi s}}{s^2 + 2s + 5} \right) \Big|_{t=\frac{5\pi}{4}} &= \frac{1}{2} e^{-\frac{\pi}{4}}. \end{aligned}$$

Question 6(a)

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ \Rightarrow A^2 &= \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \Rightarrow A^3 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} e^A &= I + A + \frac{1}{2}A^2 \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & \frac{3}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Question 6(b)

$$M = \begin{pmatrix} R \rightarrow R & F \rightarrow R & S \rightarrow R \\ R \rightarrow F & F \rightarrow F & S \rightarrow F \\ R \rightarrow S & F \rightarrow S & S \rightarrow S \end{pmatrix} = \begin{pmatrix} 0.3 & 0.7 & 0.5 \\ 0.4 & 0.2 & 0.2 \\ 0.3 & 0.1 & 0.3 \end{pmatrix}.$$

$$M^2 = \begin{pmatrix} 0.3 & 0.7 & 0.5 \\ 0.4 & 0.2 & 0.2 \\ 0.3 & 0.1 & 0.3 \end{pmatrix} \begin{pmatrix} 0.3 & 0.7 & 0.5 \\ 0.4 & 0.2 & 0.2 \\ 0.3 & 0.1 & 0.3 \end{pmatrix} = \begin{pmatrix} 0.52 & 0.4 & 0.44 \\ 0.26 & 0.34 & 0.3 \\ 0.22 & 0.26 & 0.26 \end{pmatrix}$$

$$\begin{aligned} M^3 &= \begin{pmatrix} 0.52 & 0.4 & 0.44 \\ 0.26 & 0.34 & 0.3 \\ 0.22 & 0.26 & 0.26 \end{pmatrix} \begin{pmatrix} 0.3 & 0.7 & 0.5 \\ 0.4 & 0.2 & 0.2 \\ 0.3 & 0.1 & 0.3 \end{pmatrix} \\ &= \begin{pmatrix} 0.448 & 0.448 & 0.472 \\ 0.304 & 0.28 & 0.288 \\ 0.248 & 0.232 & 0.24 \end{pmatrix} \end{aligned}$$

Answer = 0.28.

Question 7(a)

Rotate -30° , shear 45° parallel to x -axis, then rotate 30° .

$$\begin{aligned}
 T &= \begin{pmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{pmatrix} \begin{pmatrix} 1 & \tan 45 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos 30 & \sin 30 \\ -\sin 30 & \cos 30 \end{pmatrix} \\
 &= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \\
 &= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} - \frac{1}{2} & \frac{\sqrt{3}}{2} + \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \\
 &= \begin{pmatrix} 1 - \frac{\sqrt{3}}{4} & \frac{3}{4} \\ -\frac{1}{4} & 1 + \frac{\sqrt{3}}{4} \end{pmatrix}.
 \end{aligned}$$

$$T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{\sqrt{3}}{4} & \frac{3}{4} \\ -\frac{1}{4} & 1 + \frac{\sqrt{3}}{4} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ 1 + \frac{\sqrt{3}}{4} \end{pmatrix}.$$

$$\text{Answer} = \left(\frac{3}{4}, 1 + \frac{\sqrt{3}}{4} \right).$$

Question 7(b)

$$\begin{aligned}\det(A - \lambda I) &= \begin{vmatrix} 1 - \lambda & 2 \\ 2 & -2 - \lambda \end{vmatrix} = 0 \\ \implies \lambda^2 + \lambda - 6 &= 0 \\ \implies \lambda &= -3, 2.\end{aligned}$$

$$\lambda = -3,$$

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \alpha = -2.$$

$$\lambda = 2,$$

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \beta = \frac{1}{2}.$$

So

$$P = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{2} \end{pmatrix} \implies P^{-1} = \frac{2}{5} \begin{pmatrix} \frac{1}{2} & -1 \\ 2 & 1 \end{pmatrix}.$$

Hence

$$\begin{aligned}A^4 &= \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} (-3)^4 & 0 \\ 0 & 2^4 \end{pmatrix} \frac{2}{5} \begin{pmatrix} \frac{1}{2} & -1 \\ 2 & 1 \end{pmatrix} \\ &= \frac{2}{5} \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{81}{2} & -81 \\ 32 & 16 \end{pmatrix} \\ &= \begin{pmatrix} 29 & -26 \\ -26 & 68 \end{pmatrix}.\end{aligned}$$

Question 8(a)

$$\begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$$

Trace = -1 , Det = $-4 \implies$ Saddle.

$$\begin{pmatrix} 2 & -2 \\ 8 & 0 \end{pmatrix}$$

Trace = 2 , Det = $16 > 0$

$\text{Tr}^2 - 4\text{Det} = -60 < 0 \implies$ Spiral Source.

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Trace = 4 , Det = $3 > 0$

$\text{Tr}^2 - 4\text{Det} = 4 > 0 \implies$ Nodal Source.

$$\begin{pmatrix} 0 & -2 \\ 8 & 0 \end{pmatrix}$$

Trace = 0 , Det = $16 > 0$

$\text{Tr}^2 - 4\text{Det} = -64 < 0 \implies$ Centre.

Question 8(b)

Let P, S be the number of Persians and Spartans respectively.

$$\begin{aligned} \frac{dP}{dt} &= -P - \frac{15}{2}S \\ \frac{dS}{dt} &= -\frac{1}{2}P + 0 \end{aligned} \implies \begin{pmatrix} -1 & -\frac{15}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} P \\ S \end{pmatrix}.$$

Since $\det < 0$, we have a saddle.

$$\begin{aligned} \begin{vmatrix} -1 - \lambda & -\frac{15}{2} \\ -\frac{1}{2} & -\lambda \end{vmatrix} = 0 &\implies \lambda^2 + \lambda - \frac{15}{4} = 0 \\ &\implies \lambda = -\frac{5}{2}, \frac{3}{2}. \end{aligned}$$

The corresponding eigenvectors are

$$\begin{pmatrix} 1 \\ \frac{1}{5} \end{pmatrix}, \begin{pmatrix} 1 \\ -\frac{1}{3} \end{pmatrix}.$$

Using the first eigenvector, we see that as long as $S > 10000$, the solution curve will intersect the vertical axis where $P = 0$.