PC4240 : Solid State Physics 2

April 06/07 (Semester 2)

Suggested Solutions

1)

From the drift equation for charged particles in a magnetic and electric field, we can obtain the $j = \sigma E$ type of relations.

For electrons, the equation is provided in question 6.9 of Kittel.

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \frac{ne^2\tau_e / m_e}{1+\omega_c^2\tau_e^2} \begin{pmatrix} 1 & -\omega_c\tau_e & 0 \\ \omega_c\tau_e & 1 & 0 \\ 0 & 0 & 1+\omega_c^2\tau_e^2 \end{pmatrix} \tag{1}$$

For holes, we work from the drift equation in steady state i.e. $\frac{dv}{dt} = 0$.

$$v_x = \frac{e\tau_h}{m_h}E_x + \omega_c\tau_h v_y \qquad v_y = \frac{e\tau_h}{m_h}E_y - \omega_c\tau_h v_x \qquad v_z = \frac{e\tau_h}{m_h}E_z \tag{2}$$

With some manipulation, we obtain

$$v_x = \frac{e\tau_h}{m_h}E_x + \omega_c\tau_h\left(\frac{e\tau_h}{m_h}E_y - \omega_c\tau_h v_x\right) \Rightarrow \left(1+\omega_c^2\tau_h^2\right)v_x = \frac{e\tau_h}{m_h}\left(E_x + \omega_c\tau_h E_y\right)$$

$$v_y = \frac{e\tau_h}{m_h}E_y - \omega_c\tau_h\left(\frac{e\tau_h}{m_h}E_x + \omega_c\tau_h v_y\right) \Rightarrow \left(1+\omega_c^2\tau_h^2\right)v_y = \frac{e\tau_h}{m_h}\left(E_y - \omega_c\tau_h E_x\right) \tag{3}$$

$$\text{and} \qquad \left(1+\omega_c^2\tau_h^2\right)v_z = \frac{e\tau_h}{m_h}\left(1+\omega_c^2\tau_h^2\right)E_z$$

From this we form the $j = \sigma E$ relation.

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \frac{pe^2\tau_h / m_h}{1+\omega_c^2\tau_h^2} \begin{pmatrix} 1 & \omega_c\tau_h & 0 \\ -\omega_c\tau_h & 1 & 0 \\ 0 & 0 & 1+\omega_c^2\tau_h^2 \end{pmatrix} \tag{4}$$

The behavior of the semiconductor crystal is now a sum of these two contributions of electrons and holes.

Now we are more ready to tackle the question.

1a)

The Hall field is the value of E_y at the steady state situation, where $j_y = 0$.

We first find j_y from (1) and (4):

$$j_y = \frac{ne^2\tau_e}{m_e\left(1+\omega_c^2\tau_e^2\right)}\left(\omega_c\tau_e E_x + E_y\right) + \frac{pe^2\tau_h}{m_h\left(1+\omega_c^2\tau_h^2\right)}\left(-\omega_c\tau_h E_x + E_y\right) \tag{5}$$

Then we set $j_y = 0$.

$$\frac{ne^2\tau_e}{m_e\left(1+\omega_c^2\tau_e^2\right)}\left(\omega_c\tau_e E_x + E_y\right) = \frac{pe^2\tau_h}{m_h\left(1+\omega_c^2\tau_h^2\right)}\left(-E_y + \omega_c\tau_h E_x\right)$$

$$E_x\left(\frac{ne^2\tau_e^2\omega_c}{m_e\left(1+\omega_c^2\tau_e^2\right)} - \frac{pe^2\tau_h^2\omega_c}{m_h\left(1+\omega_c^2\tau_h^2\right)}\right) = E_y\left(-\frac{pe^2\tau_h}{m_h\left(1+\omega_c^2\tau_h^2\right)} - \frac{ne^2\tau_e}{m_e\left(1+\omega_c^2\tau_e^2\right)}\right)$$

Again we apply the limit $\omega_c\tau \gg k_B T$.

$$\frac{ne^2\tau_e}{m_e\left(1+\omega_{c,e}^2\tau_e^2\right)}\left(\omega_{c,e}\tau_e E_x + E_y\right) = \frac{pe^2\tau_h}{m_h\left(1+\omega_{c,h}^2\tau_h^2\right)}\left(-E_y + \omega_{c,h}\tau_h E_x\right)$$

$$E_x\left(\frac{ne^2}{m_e\omega_{c,e}} - \frac{pe^2}{m_h\omega_{c,h}}\right) = E_y\left(-\frac{pe^2}{m_h\omega_{c,h}^2\tau_h} - \frac{ne^2}{m_e\omega_{c,e}^2\tau_e}\right)$$

We then apply the relation $\omega_{c,e} = \dfrac{eB}{m_e c}, \omega_{c,h} = \dfrac{eB}{m_h c}$

and also $Q_e = \omega_{c,e}\tau_e \quad , \quad Q_h = \omega_{c,h}\tau_h$ (6)

And we are able to prove the given relation,

$$\boxed{-E_x\left(n-p\right)\left(\frac{p}{Q_h} + \frac{n}{Q_e}\right)^{-1} = E_y} \tag{7}$$

1b)

The σ_{eff} that we are trying to find occurs in $j_x = \sigma_{eff} E_x$

We can substitute (12) into (5), then factorise E_x, which would give us the required form.

We have $E_y = AE_x$ where A is the coefficient shown in (12). Thus we have:

$$j_x = \left(\sigma_{xx} + \sigma_{xy} A\right) E_x \tag{8}$$
$$\sigma_{eff} = \sigma_{xx} + \sigma_{xy} A \tag{9}$$

To find σ_{xx}, we compare the coefficients of E_x in the j_x expressions of (1) and (4).

$$\sigma_{xx} = \frac{ne^2\tau_e}{m_e\omega_{c,e}^2\tau_e^2} + \frac{pe^2\tau_h}{m_h\omega_{c,h}^2\tau_h^2}$$

$$\sigma_{xx} = e^2\left(\frac{n}{\tau_e\omega_{c,e}\, {}^{eB}\!/\!_{c}} + \frac{p}{\tau_h\omega_{c,h}\, {}^{eB}\!/\!_{c}}\right) = \frac{ec}{B}\left(\frac{n}{Q_e} + \frac{p}{Q_h}\right) \tag{10}$$

Where we have used (11) provided in the question.

Lastly, we substitute all our expressions into (14).

$$\begin{aligned}\sigma_{eff} &= \sigma_{xx} + \sigma_{xy} A \\ &= \frac{ec}{B}\left(\frac{n}{Q_e} + \frac{p}{Q_h}\right) + (-1)\frac{(n-p)ec}{B}(-1)(n-p)\left(\frac{p}{Q_h} + \frac{n}{Q_e}\right)^{-1} \\ &= \boxed{\frac{ec}{B}\left(\left(\frac{n}{Q_e} + \frac{p}{Q_h}\right) + (n-p)^2\left(\frac{n}{Q_e} + \frac{p}{Q_h}\right)^{-1}\right)}\end{aligned} \tag{11}$$

Which completes the proof.

1c)

The plot should be simple, with the two variables displaying an inverse-proportionality relationship. The only difference between the two plots would just be the coefficient.

2a)

We wish $Q = \frac{VT}{\mu_0}\left(B_c \frac{dB_c}{dT}\right)$

Consider Gibb's Free Energy, $F = U - TS$, $G = U - TS - \overrightarrow{M} \cdot \overrightarrow{H}$

For a superconductor, $G_s(B,T) - G_s(0,T) = V\frac{B^2}{2\mu_0}$

For a normal metal, $G_n(0,T) = G_n(B - B_c, T)$

Therefore we have,

$$G_s\left(B=B_c,T\right)=G_n\left(B=B_c,T\right)$$
$$=G_n\left(0,T\right)$$

Noting that at this point that there is a coexistence of states.

As a result,

$$\frac{VB_c^2}{2\mu_0}+G_s\left(0,T\right)=G_n\left(B=B_c,T\right)$$

And combining this with the previous relation,

$$G_s\left(0,T\right)=G_n\left(0,T\right)-\frac{VB_c^2}{2\mu_0}$$

Latent heat is given by $Q=T\left(S_s-S_n\right)$ with $S_s-S_n=-\frac{d\left(G_s-G_n\right)}{dT}$

And so we have,

$$Q=-T\frac{d}{dT}\left(G_s-G_n\right)$$
$$=-T\frac{d}{dT}\left(-\frac{VB_c^2}{2\mu_0}\right)$$
$$\boxed{Q=\frac{VT}{\mu_0}B_c\frac{dB_c}{dT}}$$

Since $B_c=0$ at the critical temperature, it becomes immediately clear that there is no latent heat involved at the aforementioned transition.

2b)

$$C_v=T\frac{dS}{dT}$$
$$=-T\frac{d}{dT}\left(\frac{dG}{dT}\right)$$
$$=-T\frac{d^2G}{dT^2}$$

$$
\begin{aligned}
C_{v_s} - C_{v_n} &= T\frac{d}{dT}(S_s - S_n) \\
&= T\frac{d}{dT}\left(\frac{V}{\mu_0}B_c\frac{dB}{dT}\right) \\
&= \frac{VT}{\mu_0}\left[\left(\frac{dB_c}{dT}\right)\left(\frac{dB_c}{dT}\right) + B_c\frac{d^2B_c}{dT^2}\right]_{T=T_c}
\end{aligned}
$$

$\boxed{C_{v_s} - C_{v_n} = \frac{VT}{\mu_0}\left(\frac{dB_c}{dT}\right)^2_{T=T_c}}$ (upon noting again that B_c=0 when $T = T_c$)

2c)

Don't know how to answer.

3a)

Also don't know how to answer. Really.

3b)

Domains form to minimize the demagnetization or self-energy (also called magnetostatic energy, but in the absence of a magnetic fields).

If the angle of variation between neighboring spins is small, we can make approximations in the exchange energy

$$
\begin{aligned}
E_{ex} &= -2J_{ij}S_i \cdot S_j \\
&= -2JS^2\cos\phi \\
&= -2JS^2\left(1 - \frac{1}{2}\phi^2\right) \\
&= -2JS^2 + JS^2\phi^2
\end{aligned}
$$

In which case we can see that $W_{ex} = JS^2\phi^2$ is the cost of exchange energy for twisting the neighboring spins by a small angle ϕ.

If there are N spins in the domain wall, and if the angle between the first and the last spins on the domain wall changes by 180°, then the angle between the neighboring spins is $\phi = \pi / N$.

The exchange energy cost for reversing a single spin will then be $w_{ex} = JS^2(\pi / N)^2$

The total exchange energy cost for a line of N spin is $Nw_{ex} = JS^2\pi^2 / N$
In the Bloch wall, we have planes of spins and so we are interested in the energy density per unit area (σ_{ex}) of the Bloch wall. In a square meter of a wall, there are 1/a^2 lines of spins, where a is the distance between two neighboring spins.

$\sigma_{ex} = \pi^2 JS^2 / Na^2$ (there are 1/a^2 lines per unit area)

The anisotropy energy contribution from N spins can be written as:

$$\sum_{i=1}^{N} K\sin^2\theta_i \approx \frac{N}{\pi}\int_0^{\pi} K\sin^2\theta d\theta$$
$$= NK/2$$

The anisotropy energy density per unit area of the wall is: $\sigma_{anis} = NK\frac{a}{2}$
Hence the total wall energy (per unit area) :

$$\sigma_w = \sigma_{ex} + \sigma_{anis}$$
$$= \frac{JS^2\pi^2}{Na^2} + NK\frac{a}{2}$$

The equilibrium configuration corresponds to $\frac{d\sigma_w}{dN} = 0$
Which leads to $N = \pi S\sqrt{2J/Ka^3}$
And so the width of the domain wall is $\delta = Na = \pi S\sqrt{2J/Ka}$

The energy per unit area of the domain wall is energy needed to create the domain wall:

$$\boxed{\sigma_w = \pi S\sqrt{\frac{2JK}{a}}}$$

4a)
Let ψ_1 and ψ_2 be the wavefunctions (probability amplitudes) of cooper pairs belonging to the superconductors 1 and 2 on each side of the insulator respectively.
Writing out the time dependent Schrödinger equations:

$$i\hbar\frac{d}{dt}\psi_1 = \hbar T\psi_2$$
$$i\hbar\frac{d}{dt}\psi_2 = \hbar T\psi_1$$

where T is the transfer interaction (1)

We are told in the lecture notes that the wavefunctions are:

$\psi_1 = \sqrt{n_1}e^{i\theta_1} \qquad \psi_2 = \sqrt{n_2}e^{i\theta_2}$ (2)

Using (2) in (1):

$$\frac{d}{dt}\psi_1 = \frac{1}{2}\left(\sqrt{n_1}\right)^{-1} e^{i\theta_1}\frac{dn_1}{dt} + i\psi_1\frac{d\theta_1}{dt} = -iT\psi_2 \qquad (3)$$

$$\frac{d}{dt}\psi_2 = \frac{1}{2}\left(\sqrt{n_2}\right)^{-1} e^{i\theta_2}\frac{dn_2}{dt} + i\psi_2\frac{d\theta_2}{dt} = -iT\psi_1 \qquad (4)$$

Multiplying (3) by $\sqrt{n_1}e^{-i\theta_1}$:

$$\frac{1}{2}\frac{dn_1}{dt} + in_1\frac{d\theta_1}{dt} = -iT\left(n_1 n_2\right)^{1/2} e^{i\delta} \quad \text{with } \delta = \theta_1 - \theta_2 \qquad (5)$$

Multiplying (4) by $\sqrt{n_2}e^{-i\theta_2}$

$$\frac{1}{2}\frac{dn_2}{dt} + in_2\frac{d\theta_2}{dt} = -iT\left(n_1 n_2\right)^{1/2} e^{-i\delta} \qquad (6)$$

Equating the real and imaginary parts in (5) and (6):

$$\frac{dn_1}{dt} = 2T\left(n_1 n_2\right)^{1/2}\sin\delta \;\; ; \;\; \frac{dn_2}{dt} = -2T\left(n_1 n_2\right)^{1/2}\sin\delta \qquad (7)$$

$$\frac{d\theta_1}{dt} = -T\sqrt{\frac{n_1}{n_2}}\cos\delta \quad ; \quad \frac{d\theta_2}{dt} = -T\sqrt{\frac{n_1}{n_2}}\cos\delta \qquad (8)$$

if $n_1 \approx n_2$,

$$\frac{d\theta_1}{dt} = \frac{d\theta_2}{dt} \rightarrow \frac{d}{dt}\left(\theta_1 - \theta_2\right) = 0 \qquad (9)$$

$$\frac{dn_1}{dt} = -\frac{dn_2}{dt} \qquad (10)$$

The current density

$$J \propto \frac{dn_2}{dt} \propto -\frac{dn_1}{dt}$$

$$J \propto \sin\delta$$

$$\Rightarrow \boxed{J = J_0 \sin\left(\theta_1 - \theta_2\right)}$$

4b)

As compared to 4(a), the Schrödinger equations differ by an extra term:

$$i\hbar\frac{d}{dt}\psi_1 = \hbar T\psi_2 - eV\psi_1$$

$$i\hbar\frac{d}{dt}\psi_2 = \hbar T\psi_1 + eV\psi_2$$

Performing the same calculations:

$$\frac{dn_2}{dt} = -2T\sqrt{n_1 n_2}\,\sin\delta$$

$$\frac{dn_1}{dt} = 2T\sqrt{n_1 n_2}\,\sin\delta$$

$$\frac{d\theta_2}{dt} = -\left(\frac{eV}{\hbar}\right) - T\cos\delta\sqrt{\frac{n_1}{n_2}}$$

$$\frac{d\theta_1}{dt} = \left(\frac{eV}{\hbar}\right) - T\cos\delta\sqrt{\frac{n_2}{n_1}}$$

Under the same case where $n_1 \approx n_2$:

$$\frac{d}{dt}(\theta_1 - \theta_2) = \frac{d}{dt}\delta(t) = -\frac{2eV}{\hbar}$$

$$\Rightarrow \delta(t) = \delta(0) - \frac{2eVt}{\hbar}$$

$$J = J_0 \sin\delta(t)$$

$$\Rightarrow \boxed{J = J_0 \sin\left[\delta(0) - \frac{2eVt}{\hbar}\right]}$$

4c)

The direct current SQUID (Superconducting Quantum Interference Device) consists of two Josephson junctions arranged on a superconducting ring. A current applied to the SQUID, called a bias current, divides between the junctions and, if greater than the critical current, produces a voltage across the SQUID. Plotting this current against the voltage yields characteristic curves. Steadily increasing the magnetic flux threading through the ring (e.g. bringing in a small magnet) causes the critical current to decrease and then increase successively. The critical current is a maximum for zero flux (for an integer number of flux quanta) and a minimum for a half-integer number of flux quanta. The period of these oscillations is the flux quantum.