PC4246 Quantum Optics AY2012/2013

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Question 1

1.

$$\begin{aligned}
a|\alpha\rangle &= e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}a|n\rangle \\
&= e^{-|\alpha|^2/2}\sum_{n=1}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\sqrt{n}|n-1\rangle \\
&= \alpha e^{-|\alpha|^2/2}\sum_{n=1}^{\infty}\frac{\alpha^{n-1}}{\sqrt{(n-1)!}}|n-1\rangle \\
&= \alpha e^{-|\alpha|^2/2}\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}|m\rangle \\
&= \alpha|\alpha\rangle
\end{aligned}$$

2. Note that $\hat{D}(\alpha)$ is a unitary operator. We have the following relations:

$$D^\dagger(\alpha)aD(\alpha) = a+\alpha$$

Acting on vacuum, we have

$$D^\dagger(\alpha)aD(\alpha)|0\rangle = a|0\rangle+\alpha|0\rangle = \alpha|0\rangle$$

Then, using the unitarity of $D(\alpha)$, we get

$$aD(\alpha)|0\rangle = \alpha D(\alpha)|0\rangle$$

Therefore $D(\alpha)|0\rangle$ is an eigenstate of a, i.e. $D(\alpha)|0\rangle = |\alpha\rangle$

3.

$$\begin{aligned}
\langle\alpha|Q^2|\alpha\rangle &= \langle\alpha|(a^\dagger+a)^2|\alpha\rangle \\
&= \langle\alpha|((a^\dagger)^2+a^\dagger a+aa^\dagger+a^2)|\alpha\rangle \\
&= (\alpha^*)^2+2\alpha^*\alpha+1+\alpha^2
\end{aligned}$$

$$\langle\alpha|Q|\alpha\rangle^2 = (\alpha^* + \alpha)^2$$

$$\Delta Q_\alpha = 1$$

Similarly, $\Delta P_\alpha = 1$.

4.

$$\begin{aligned} p_m &= e^{-|\alpha|^2} \frac{\alpha^{2n}}{n!} \\ &= e^{-\bar{n}} \frac{\bar{n}^n}{n!} \end{aligned}$$

Question 2

1. (a)

Coherent state $|\alpha\rangle$:

$$g^{(2)}(0) = \frac{\alpha^{*2}\alpha^2}{|\alpha|^4} = 1$$

Number state $|n\rangle$:
Fock state with n photons: $\langle a^\dagger a\rangle^2 = n^2$and

$$a^{\dagger 2}a^2|n\rangle = (n-1)n|n\rangle,$$

whence $g^{(2)}(0) = \frac{n-1}{n}$.

Thermal state ρ_T:
We know that $\langle a^\dagger a\rangle = \bar{n} = \frac{x}{1-x}$ with $x = e^{-\beta\hbar\omega}$; using $a^{\dagger 2}a^2|n\rangle = (n-1)n|n\rangle$, we have

$$Tr(\rho_T a^{\dagger 2}a^2) = \sum_n p_n n(n-1) = (1-x)\sum_n x^n n(n-1) = (1-x)x^2 \sum_n \frac{d^2}{dx^2}x^n = 2(\frac{x}{1-x})^2$$

whence $g^{(2)}(0) = 2$.

An arbitrary mixed state $\rho = \sum_{n,m=0}^{\infty} \rho_{n,m}|n\rangle\langle m|$:

$$\begin{aligned} Tr(a^{\dagger 2}a^2\rho) &= Tr[\sum_{n,m=2}^{\infty} \rho_{n,m}\sqrt{nm}\sqrt{(n-1)(m-1)}|n-2\rangle\langle m-2|] \\ &= \sum_{n=1}^{\infty} \sqrt{n^2\sqrt{(n+1)^2}} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \rho_n n(n+1) \\
Tr(a^\dagger a\rho) &= \sum_{n,m=1}^{\infty} \rho_{n,m} |n-1\rangle\langle m-1| \sqrt{nm} \\
&= \sum_{n=1}^{\infty} \rho_n n \\
g^{(2)}(0) &= \frac{Tr(a^{\dagger 2} a\rho)}{[Tr(a^\dagger a\rho)]^2} \\
&= \frac{\sum_{n=1}^{\infty} \rho_n n(n+1)}{(\sum_{n=1}^{\infty} \rho_n n)^2}
\end{aligned}
$$

1. (b)

Denote $|\phi\rangle = a|\psi\rangle = ||a|\psi\rangle|||\psi\rangle$ an unnormalised state. Then we have $\langle\psi|a^{\dagger 2}a^2|\psi\rangle = \langle\phi|a^\dagger a|\phi\rangle = ||a|\psi\rangle||^2\langle\psi|a^\dagger a|\psi\rangle$, which is the desired result.
This means that the coincidence rate can be obtained by first measuring one photon, then measuring another one on the updated state (i.e., on the state obtained from the initial one by removing a photo).

2.

$$
\vec{E}_1^{(+)}\vec{E}_2^{(+)}|\psi\rangle = \int d\underline{\omega} e^{i(\omega t_1 - kZ_1)} e^{i(\omega' t_2 - k' Z_2)} \psi(\underline{\omega}) a_1(\omega) a_2(\omega') a_1^\dagger(\omega'') a_2^\dagger(\omega''')|0\rangle
$$

where $d\underline{\omega} = d\omega d\omega' d\omega'' d\omega'''$

$$
\begin{aligned}
a_1(\omega)a_2(\omega')a_1^\dagger(\omega'')a_2^\dagger(\omega''')|0\rangle &= (\delta(\omega-\omega'') - a_1^\dagger(\omega'')a_1(\omega))(\delta(\omega'-\omega''') - a_z^\dagger(\omega''')a_z(\omega'))|0\rangle \\
&= \delta(\omega-\omega'')\delta(\omega'-\omega''')|0\rangle
\end{aligned}
$$

so,

$$
\begin{aligned}
\vec{E}_1^{(+)}\vec{E}_2^{(+)}|\psi\rangle &= \int d\omega d\omega' e^{i(\omega t_1 - kZ_1)} e^{i(\omega' t z_1 - kZ_2)}|0\rangle \\
&= g(x_1, t_1, x_z t_z)|0\rangle
\end{aligned}
$$

where $g(x,t) = \int d\omega e^{i(\omega t - kz)} \psi(\omega)$

$$
P_{delection} = |g|^2
$$

Question 3

1. is trivial

2. At resonance:

$$|n,\pm\rangle = \frac{1}{\sqrt{2}}\left\{|n,g\rangle \pm |n-1,e\rangle\right\}$$

, and so:

$$|n,g\rangle = \frac{1}{\sqrt{2}}\left(|n,+\rangle + |n,-\rangle\right)$$

The evolution is then

$$\begin{aligned}
|\psi_{(t)}\rangle &= e^{-itH/\hbar}|n,g\rangle \\
&= \frac{1}{\sqrt{2}}\left(e^{-itE_+/\hbar}|n,+\rangle + e^{itE_-/\hbar}|n,-\rangle\right) \\
&= \frac{e^{-itn\omega}}{\sqrt{2}}\left[\frac{2\cos(tg\sqrt{n})}{\sqrt{2}}|n,g\rangle - \frac{2i\sin(g\sqrt{n}t)}{\sqrt{2}}|n-1,e\rangle\right]
\end{aligned}$$

Therefore

$$P_g(t) = \cos^2(g\sqrt{n}t) = \frac{1}{2}\left[1+\cos(2g\sqrt{n}t)\right]$$