Characterizing Genuine Multilevel Entanglement: an Alternative Approach

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To my parents

To my supervisor – Professor KWEK LEONG CHUAN

To Professor VALERIO SCARANI

To Professor LEE SOO TECK

To Professor TO WING KEUNG

Thank you for your continued guidance and support.

Abstract

In this report, we discuss an alternative method to characterize 'decomposability' as defined in [1]. While the authors in that paper relied on a numerical technique, we shall demonstrate an algebraic approach. We also extend our result to general $D \times D$ bipartite systems, and further consider multipartite systems and 'finer' decompositions.

Contents

1 Introduction

Let us consider a thought experiment. Suppose Alice and Bob have the maximally entangled two-ququart state, $|\psi\rangle = \frac{1}{2}(|00\rangle + |11\rangle + |22\rangle + |33\rangle)$, where for brevity, we have denoted $|i\rangle_A \otimes |i\rangle_B$ by $|ii\rangle$. Feeling particularly inspired one afternoon, they decide to prepare two pairs of maximally entangled qubits, $|\varphi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{A_1B_1}$ and $|\varphi'\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{A_2B_2}$.

Consider the following sequence of operations:

$$\begin{aligned}
|\varphi\rangle \otimes |\varphi'\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{A_1B_1} \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{A_2B_2} \\
&= \frac{1}{2}(|00\rangle_{A_1B_1} |00\rangle_{A_2B_2} + |00\rangle_{A_1B_1} |11\rangle_{A_2B_2} + \\
&\quad |11\rangle_{A_1B_1} |00\rangle_{A_2B_2} + |11\rangle_{A_1B_1} |11\rangle_{A_2B_2}) \\
&= \frac{1}{2}(|00\rangle_{A_1A_2} |00\rangle_{B_1B_2} + |01\rangle_{A_1A_2} |01\rangle_{B_1B_2} + \\
&\quad |10\rangle_{A_1A_2} |10\rangle_{B_1B_2} + |11\rangle_{A_1A_2} |11\rangle_{B_1B_2}).
\end{aligned}$$

Now if we regard the qubits A_1 and A_2 collectively as a system in itself, we have a ququart system spanned by the orthonormal set

$$\{|00\rangle_{A_1A_2}, |01\rangle_{A_1A_2}, |10\rangle_{A_1A_2}, |11\rangle_{A_1A_2}\}.$$

Making the identification $|00\rangle_{A_1A_2} \mapsto |0\rangle_A, |01\rangle_{A_1A_2} \mapsto |1\rangle_A, |10\rangle_{A_1A_2} \mapsto |2\rangle_A$ and $|11\rangle_{A_1A_2} \mapsto |3\rangle_A$ for Alice, and likewise for Bob, we observe that $|\psi\rangle = |\varphi\rangle \otimes |\varphi'\rangle$, i.e. the maximally entangled state appears to be put into a separable form, namely a tensor product of $|\varphi\rangle$ and $|\varphi'\rangle$.

What have we done here? It is perhaps best to first provide an illustration.

From Figure 1, which we hope is somewhat self-explanatory, Alice and Bob had prepared two systems, the maximally entangled qubits A_1B_1 and A_2B_2.

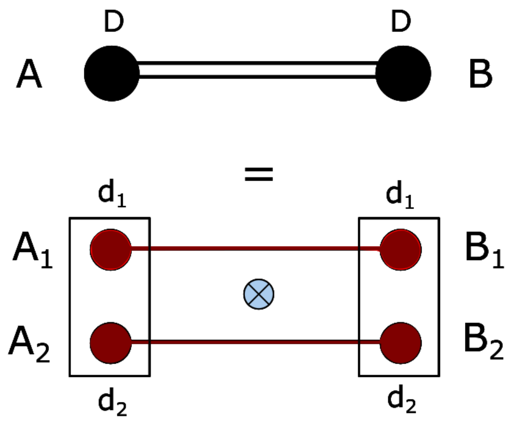


Figure 1: Simulating a pair of entangled ququarts, where $|\psi\rangle = \frac{1}{2}(|00\rangle + |11\rangle + |22\rangle + |33\rangle)$.

Taking the tensor product of these two systems, we have a 4-qubit composite system. At this point, Alice and Bob rename their composite system (this is represented by the 'swapping' mentioned above), such that the order becomes $A_1A_2B_1B_2$ (before renaming, it was $A_1B_1A_2B_2$). Noting that A_1A_2 and B_1B_2 are 4-dimensional systems, they could be used in place of the ququarts A and B, thus the 'identification'. In other words, the system A_1A_2 simulates A, and the system B_1B_2 simulates B.

This opens a wide range of questions. Under what conditions can a composite system AB be simulated with lower-dimensional systems? How do we detect these conditions? Can we do the same for n-partite systems, as opposed to just bipartite systems (of which AB is an example)? Can we create 'finer' simulations, i.e. for suitably large dimensions of A and B, can we simulate A with not just A_1A_2, but $A_1A_2...A_n$ with $n > 2$?

These questions are important, because high-dimensional entanglement are hard to create experimentally, and simulating them with lower-dimensional systems which are comparatively easier to prepare may come in very handy for the experimentalists. In this paper, we shall explore these questions, though not always ending on a high note.

Before we further proceed, let us briefly digress and review the mathematical tools needed.

2 Mathematical Tools

For ease of reference and completeness, we shall in this chapter discuss in detail the mathematics we need. We start with the matrix representation of tensor products, which while conceptually simple provides much needed notational convenience. Next, we move on to the Schmidt decomposition, which has become a ubiquitous tool in quantum information theory. We shall prove it 'from scratch'. Finally, we discuss the concept of permutation matrices, which turned out to be the perfect tool to elucidate a combinatorial result.

2.1 Matrix Representation of Tensor Products

Let V_1, V_2, W_1 and W_2 be complex vector spaces of dimensions n, m, n' and m' respectively, with bases $B_{V_1} = \{e_1, \dots, e_n\}, B_{V_2} = \{f_1, \dots, f_m\}, B_{W_1} = \{\tilde{e}_1, \dots, \tilde{e}_{n'}\}$ and $B_{W_2} = \{\tilde{f}_1, \dots, \tilde{f}_{m'}\}$.

2.1.1 Vectors

Let us have the arbitrary vectors

$$|v_1\rangle \in V_1 \xmapsto{B_{V_1}} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \quad |v_2\rangle \in V_2 \xmapsto{B_{V_2}} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix},$$

$$|w_1\rangle \in W_1 \xmapsto{B_{W_1}} \begin{bmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_{n'} \end{bmatrix}, \quad |w_2\rangle \in W_2 \xmapsto{B_{W_2}} \begin{bmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_{m'} \end{bmatrix}.$$

where the symbol $\xmapsto{B}$ refers to the vector being represented in column form with respect to the basis B.

Proposition 2.1.1. Choose the ordered bases for $V_1 \otimes V_2$ and $W_1 \otimes W_2$ as follows:

$$\begin{aligned} B_{V_1 \otimes V_2} = \{&e_1 \otimes f_1, \ldots, e_1 \otimes f_m, \\ &e_2 \otimes f_1, \ldots, e_2 \otimes f_m, \\ &\ldots \\ &e_n \otimes f_1, \ldots, e_n \otimes f_m\} \end{aligned}$$

and

$$\begin{aligned} B_{W_1 \otimes W_2} = \{&\tilde{e}_1 \otimes \tilde{f}_1, \ldots, \tilde{e}_1 \otimes \tilde{f}_{m'}, \\ &\tilde{e}_2 \otimes \tilde{f}_1, \ldots, \tilde{e}_2 \otimes \tilde{f}_{m'}, \\ &\ldots \\ &\tilde{e}_{n'} \otimes \tilde{f}_1, \ldots, \tilde{e}_{n'} \otimes \tilde{f}_{m'}\}. \end{aligned}$$

Then we have

$$|v_1\rangle \otimes |v_2\rangle \in V_1 \otimes V_2 \overset{B_{V_1 \otimes V_2}}{\longmapsto} \begin{bmatrix} a_1 \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \\ \vdots \\ a_n \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_1 b_1 \\ \vdots \\ a_1 b_m \\ \vdots \\ a_n b_1 \\ \vdots \\ a_n b_m \end{bmatrix}$$

and

$$|w_1\rangle \otimes |w_2\rangle \in W_1 \otimes W_2 \overset{B_{W_1 \otimes W_2}}{\longmapsto} \begin{bmatrix} \tilde{a}_1 \begin{bmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_{m'} \end{bmatrix} \\ \vdots \\ \tilde{a}_{n'} \begin{bmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_{m'} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \tilde{a}_1 \tilde{b}_1 \\ \vdots \\ \tilde{a}_1 \tilde{b}_{m'} \\ \vdots \\ \tilde{a}_{n'} \tilde{b}_1 \\ \vdots \\ \tilde{a}_{n'} b_{m'} \end{bmatrix}.$$

Proof. The proof is straightforward. Expanding $|v_1\rangle \otimes |v_2\rangle$ in our chosen *ordered* basis, we have

$$\begin{aligned} |v_1\rangle \otimes |v_2\rangle &= (\sum_{i=1}^{n} a_i e_i) \otimes (\sum_{j=1}^{m} b_j f_j) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j e_i \otimes f_j \overset{B_{V_1 \otimes V_2}}{\longmapsto} \begin{bmatrix} \tilde{a}_1 \tilde{b}_1 \\ \vdots \\ \tilde{a}_1 \tilde{b}_{m'} \\ \vdots \\ \tilde{a}_{n'} \tilde{b}_1 \\ \vdots \\ \tilde{a}_{n'} b_{m'} \end{bmatrix}. \end{aligned}$$

and likewise for $|w_1\rangle \otimes |w_2\rangle$. □

Remark 2.1.2. Note that the choice of ordered basis is important. Our choice is standard in the literature, and almost always implicitly assumed. But this choice is not canonical - suppose we had chosen the equally 'natural' bases

$$\begin{aligned} B'_{V_1 \otimes V_2} = \{ & e_1 \otimes f_1, \ldots, e_n \otimes f_1, \\ & e_1 \otimes f_2, \ldots, e_n \otimes f_2, \\ & \cdots \\ & e_1 \otimes f_m, \ldots, e_n \otimes f_m \} \end{aligned}$$

and

$$\begin{aligned} B'_{W_1 \otimes W_2} = \{ & \tilde{e}_1 \otimes \tilde{f}_1, \ldots, \tilde{e}_{n'} \otimes \tilde{f}_1, \\ & \tilde{e}_1 \otimes \tilde{f}_2, \ldots, \tilde{e}_{n'} \otimes \tilde{f}_2, \\ & \cdots \\ & \tilde{e}_1 \otimes \tilde{f}_{m'}, \ldots, \tilde{e}_{n'} \otimes \tilde{f}_{m'} \} \end{aligned}$$

instead. Then we would have

$$|v_1\rangle \otimes |v_2\rangle \overset{B'_{V_1\otimes V_2}}{\longmapsto} \begin{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} b_1 \\ \vdots \\ \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} b_m \end{bmatrix}, \quad |w_1\rangle \otimes |w_2\rangle \overset{B'_{W_1\otimes W_2}}{\longmapsto} \begin{bmatrix} \begin{bmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_{n'} \end{bmatrix} \tilde{b}_1 \\ \vdots \\ \begin{bmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_{n'} \end{bmatrix} \tilde{b}_{m'} \end{bmatrix}.$$

Later on, we shall find the need to compare the elements in this representation. We will introduce a way of rearranging this representation so as to make the comparison more convenient. It turns out that this rearrangement allows for a precise and succinct mathematical formulation of our problem.

2.1.2 Linear Maps

Let us have the linear maps $T_1 : V_1 \longrightarrow W_1$ and $T_2 : V_2 \longrightarrow W_2$. Denote the matrix representations of T_1 and T_2 with respect to the given bases as follows:

$$[T_1]_{B_{V_1}}^{B_{W_1}} = [\alpha_{ij}] = \begin{bmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n'1} & \dots & \alpha_{n'n} \end{bmatrix}, \quad [T_2]_{B_{V_2}}^{B_{W_2}} = [\beta_{ij}] = \begin{bmatrix} \beta_{11} & \dots & \beta_{1m} \\ \vdots & \ddots & \vdots \\ \beta_{m'1} & \dots & \beta_{m'm} \end{bmatrix}.$$

The **tensor product of the linear maps** T_1 and T_2 is defined as the following linear map:

$$\begin{aligned} T_1 \otimes T_2 : \quad & V_1 \otimes V_2 \longrightarrow W_1 \otimes W_2 \\ & \sum_i c_i v_{1i} \otimes v_{2i} \longmapsto \sum_i c_i T_1(v_{1i}) \otimes T_2(v_{2i}). \end{aligned}$$

Choose the ordered bases for $V_1 \otimes V_2$ and $W_1 \otimes W_2$ as defined in Proposition 2.1.1. Since

$$(T_1 \otimes T_2)(e_k \otimes f_l) = \sum_{i=1}^{n'} \sum_{j=1}^{m'} \alpha_{ik}\beta_{jl}(\tilde{e}_i \otimes \tilde{f}_j),$$

we have

$$[T_1 \otimes T_2]_{B_{V_1 \otimes V_2}}^{B_{W_1 \otimes W_2}} = [\alpha_{ij}[\beta_{kl}]] = \begin{bmatrix} \alpha_{11}[\beta_{kl}] & \dots & \alpha_{1n}[\beta_{kl}] \\ \vdots & \ddots & \vdots \\ \alpha_{n'1}[\beta_{kl}] & \dots & \alpha_{n'n}[\beta_{kl}] \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_{11} \begin{bmatrix} \beta_{11} & \dots & \beta_{1m} \\ \vdots & \ddots & \vdots \\ \beta_{m'1} & \dots & \beta_{m'm} \end{bmatrix} & \dots & \alpha_{1n} \begin{bmatrix} \beta_{11} & \dots & \beta_{1m} \\ \vdots & \ddots & \vdots \\ \beta_{m'1} & \dots & \beta_{m'm} \end{bmatrix} \\ \vdots & \ddots & \vdots \\ \alpha_{n'1} \begin{bmatrix} \beta_{11} & \dots & \beta_{1m} \\ \vdots & \ddots & \vdots \\ \beta_{m'1} & \dots & \beta_{m'm} \end{bmatrix} & \dots & \alpha_{n'n} \begin{bmatrix} \beta_{11} & \dots & \beta_{1m} \\ \vdots & \ddots & \vdots \\ \beta_{m'1} & \dots & \beta_{m'm} \end{bmatrix} \end{bmatrix}.$$

Remark 2.1.3. Again, this choice of ordered basis is arbitrary, but standard.

2.1.3 Separability of Column Vectors/Matrices

A (suitably-sized) column vector **v** is said to be **separable** if there exist column vectors **a** and **b** such that

$$\mathbf{v} = \mathbf{a} \otimes \mathbf{b} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

Similarly, a (suitably-sized) matrix **M** is said to be **separable** if there exist matrices **A** and **B** such that

$$\mathbf{M} = \mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n'1} & \dots & \alpha_{n'n} \end{bmatrix} \otimes \begin{bmatrix} \beta_{11} & \dots & \beta_{1m} \\ \vdots & \ddots & \vdots \\ \beta_{m'1} & \dots & \beta_{m'm} \end{bmatrix}.$$

2.2 The Schmidt Decomposition

2.2.1 Singular Value Decomposition

Definition 2.2.1. Let $T : V \longrightarrow W$ be a linear transformation, where V and W are finite-dimensional inner product spaces with inner products $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$ respectively. A function $T^\dagger : W \longrightarrow V$ is called an **adjoint** of T if $\langle T(x), y \rangle_W = \langle x, T^\dagger(y) \rangle_V$ for all $x \in V$ and $y \in W$.

Lemma 2.2.2.

1. $T^\dagger T$ and $TT^\dagger$ are positive semidefinite.

2. $\text{rank}(T^\dagger T) = \text{rank}(TT^\dagger) = \text{rank}(T)$.

Proof. Refer to pages 367 and 378 of [2]. □

Theorem 2.2.3 (Singular Value Theorem for Linear Transformations)**.** Let V and W be finite-dimensional inner product spaces of dimensions n and m respectively, and let $T : V \longrightarrow W$ be a linear transformation of rank r, i.e. $\dim(T(V)) = r$. (Note that $r \leq \min(n, m)$). Then

1. There exist orthonormal bases $\{v_1, v_2, \ldots, v_n\}$ for V and $\{w_1, w_2, \ldots, w_m\}$ for W, and *positive* scalars $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ such that

$$T(v_i) = \begin{cases} \sigma_i w_i & \text{if } i \leq r \\ 0 & \text{if } i > r. \end{cases} \tag{1}$$

2. Furthermore, for $1 \leq i \leq n$, v_i is an eigenvector of $T^\dagger T$ with corresponding eigenvalues σ_i^2 if $i \leq r$ and 0 if $i > r$. Thus the scalars $\{\sigma_i\}$, called the **singular values** of T, are uniquely determined by T.

Proof.

1. From Lemma 2.2.2, $T^\dagger T$ is a positive semidefinite linear operator of rank r on V, hence there is an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ for V comprising eigenvectors of $T^\dagger T$ with corresponding eigenvalues λ_i, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$, and $\lambda_i = 0$ for $i > r$. For $i \leq r$, define $\sigma_i = \sqrt{\lambda_i}$ and $w_i = \frac{1}{\sigma_i} T(v_i)$. We show that $\{w_1, w_2, \ldots, w_r\}$ forms an orthonormal subset of W. Suppose $1 \leq i, j \leq r$. Then
$$\begin{aligned} \langle w_i, w_j \rangle &= \langle \frac{1}{\sigma_i} T(v_i), \frac{1}{\sigma_j} T(v_j) \rangle \\ &= \frac{1}{\sigma_i \sigma_j} \langle T^\dagger T(v_i), v_j \rangle \\ &= \frac{1}{\sigma_i \sigma_j} \langle \lambda_i v_i, v_j \rangle \\ &= \frac{\sigma_i^2}{\sigma_i \sigma_j} \langle v_i, v_j \rangle = \delta_{ij}. \end{aligned}$$
This set can be extended to an orthonormal basis $\{w_1, \ldots, w_r, \ldots, w_m\}$ for W. Now we prove Equation 1. By definition, $T(v_i) = \sigma_i w_i$ for $i \leq r$. If $i > r$, $T^\dagger T(v_i) = 0$, so $T(v_i) = 0$ (c.f. *pg* 367, [2]).

2. For $1 \leq i \leq m$ and $1 \leq j \leq n$,
$$\langle T^\dagger(w_i), v_j \rangle = \langle w_i, T(v_j) \rangle = \begin{cases} \sigma_i & \text{if } i = j \leq r \\ 0 & \text{otherwise} \end{cases},$$
so
$$T^\dagger(w_i) = \sum_{j=1}^{n} \langle T^\dagger(w_i), v_j \rangle v_j = \begin{cases} \sigma_i v_i & \text{if } i = j \leq r \\ 0 & \text{otherwise.} \end{cases}$$
Thus for $i \leq r$,
$$T^\dagger T(v_i) = T^\dagger(\sigma_i w_i) = \sigma_i^2 w_i$$
and for $i > r$, $T^\dagger T(v_i) = T^\dagger(0) = 0$. Therefore each v_i is an eigenvector of $T^\dagger T$ with corresponding eigenvalue σ_i^2 if $i \leq r$ and 0 if $i > r$.

□

Remark 2.2.4. Although the singular values $\{\sigma_i\}$ are unique, the orthonormal bases $\{v_i\}$ and $\{w_i\}$ are not, because the orthonormal basis formed by the eigenvectors of $T^\dagger T$ is not unique.

Theorem 2.2.5 (Singular Value Decomposition for Matrices). Let A be an $m \times n$ complex-valued matrix of rank r. There exists an $m \times m$ unitary matrix U, an $n \times n$ unitary matrix V, and an $m \times n$ matrix Σ defined by

$$\Sigma_{ij} = \begin{cases} \sigma_i & \text{if } i = j \leq r \\ 0 & \text{otherwise} \end{cases}$$

where $\sigma_i > 0$, such that

$$A = U \Sigma V^\dagger.$$

Proof. Define $L_A : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ such that $L_A(x) = Ax$, where $x \in \mathbb{C}^n$. By Theorem 2.2.3, there exist orthonormal bases $\beta = \{v_1, v_2, \ldots, v_n\}$ for $\mathbb{C}^n$ and $\gamma = \{w_1, w_2, \ldots, w_m\}$ for $\mathbb{C}^m$ and scalars $\{\sigma_1, \ldots \sigma_r\}$ such that $L_A(v_i) = \sigma_i w_i$ for $i \leq r$ and 0 for $i > r$.

Let U be the $m \times m$ matrix whose jth column is w_j for all j, and V be the $n \times n$ matrix whose jth column is v_j for all j. Note that because of orthonormality, U and V are both unitary. Now consider the jth column of the matrices AV and $U\Sigma$, where Σ is formed from the scalars $\{\sigma_1, \ldots \sigma_r\}$ and defined as in the statement of the theorem. It is straightforward to show that both jth columns are $\sigma_j w_j$. Thus AV and $U\Sigma$ are equal, implying $A = U\Sigma V^\dagger$. □

Remark 2.2.6. The singular values of a matrix A are defined to be the singular values of L_A as defined above.

2.2.2 Schmidt from SVD

The Schmidt Decomposition follows from the Singular Value Decomposition.

Theorem 2.2.7 (Schmidt Decomposition). Suppose we have a bipartite pure state

$$|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B.$$

Then it is possible to express this state as follows:

$$|\psi\rangle_{AB} = \sum_{i=0}^{d-1} \lambda_i \, |i\rangle_A \, |i\rangle_B \,,$$

where the coefficients λ_i are real, strictly positive and satisfy $\sum_i \lambda_i^2 = 1$, the states $\{|i\rangle_A\}$ form an orthonormal basis for system A and the states $\{|i\rangle_B\}$ form an orthonormal basis for system B. The *Schmidt rank* d satisfies

$$d \leq \min\{\dim(\mathcal{H}_A), \dim(\mathcal{H}_B)\}.$$

Proof. Denoting $d_A = \dim(\mathcal{H}_A)$ and $d_B = \dim(\mathcal{H}_B)$, we can express $|\psi\rangle_{AB}$ as

$$|\psi\rangle_{AB} = \sum_{j=0}^{d_A-1} \sum_{k=0}^{d_B-1} a_{jk} \, |j\rangle_A \, |k\rangle_B \tag{2}$$

for complex coefficients a_{jk} and orthonormal bases $|j\rangle_A$ and $|k\rangle_B$. Consider the matrix A defined by

$$[A]_{jk} = a_{jk}.$$

Making use of Theorem 2.2.5, we write A as

$$A = U \Sigma V$$

where U is a $d_A \times d_A$ unitary matrix, V is a $d_B \times d_B$ unitary matrix and Σ is a $d_A \times d_B$ matrix with d real, strictly positive numbers λ_i along its diagonal and zeroes elsewhere. Thus for $1 \leq j \leq d_A, 1 \leq k \leq d_B$, we have

$$a_{jk} = \sum_{i=0}^{d-1} u_{ji} \lambda_i v_{ik}.$$

Substituting this into Equation 2, we have

$$
\begin{aligned}
|\psi\rangle_{AB} &= \sum_{j=0}^{d_A-1}\sum_{k=0}^{d_B-1}\left(\sum_{i=0}^{d-1} u_{ji}\lambda_i v_{ik}\right)|j\rangle_A |k\rangle_B \\
&= \sum_{i=0}^{d-1}\lambda_i\left(\sum_{j=0}^{d_A-1} u_{ji}\,|j\rangle_A\right)\otimes\left(\sum_{k=0}^{d_B-1} v_{ik}\,|k\rangle_B\right) \\
&= \sum_{i=0}^{d-1}\lambda_i\,|i\rangle_A |i\rangle_B\,,
\end{aligned}
$$

where

$$|i\rangle_A = \sum_j u_{ji}\,|j\rangle_A \tag{3}$$

$$|i\rangle_B = \sum_k v_{ik}\,|k\rangle_B\,. \tag{4}$$

Now we verify that $\{|i\rangle_A\}_0^{d_A-1}$ and $\{|i\rangle_B\}_0^{d_B-1}$ form orthonormal bases for their respective systems, and that $\sum_i \lambda_i^2 = 1$.

$$
\begin{aligned}
{}_A\langle i' \,|\, i\rangle_A &= \left(\sum_j u^*_{ji'}\,\langle j|_A\right)\left(\sum_k u_{ki}\,|k\rangle_A\right) \\
&= \sum_j\sum_k u^*_{ji'}u_{ki}\;{}_A\langle j \,|\, k\rangle_A \\
&= \sum_j\sum_k u^*_{ji'}u_{ki}\delta_{jk} \\
&= \sum_j u^*_{ji'}u_{ji} \\
&= \sum_j U^\dagger_{i'j}U_{ji} = \delta_{i'i}.
\end{aligned}
$$

Likewise for ${}_B\langle j' \,|\, j \rangle_B$. Finally, since $\sum_j \sum_k |a_{jk}|^2 = 1$, we have

$$\begin{aligned}
1 &= \sum_j \sum_k \left(\sum_p u_{jp} \lambda_p v_{pk} \right) \left(\sum_q u^*_{jq} \lambda_q v^*_{qk} \right) \\
&= \sum_j \sum_k \sum_p \sum_q U_{jp} U^{-1}_{qj} \lambda_p \lambda_q V_{pk} V^{-1}_{kq} \\
&= \sum_p \sum_q \lambda_p \lambda_q \left(\sum_j U^{-1}_{qj} U_{jp} \right) \left(\sum_k V^{-1}_{kq} V_{pk} \right) \\
&= \sum_p \sum_q \delta^2_{pq} \lambda_p \lambda_q \\
&= \sum_p \lambda^2_p,
\end{aligned}$$

finishing the proof. □

2.2.3 The Schmidt CoB Matrix is Separable!

This result is important later on when we prove that 'decomposability' is still preserved after we make a change-of-basis from an initial basis to a Schmidt Basis. First, we recall the notion of a change-of-basis (CoB) matrix.

Given a linear operator $T : V \longrightarrow V$. Let us adopt two bases for V, say $B = \{e_1, e_2, \dots, e_n\}$ for the domain and $B' = \{e'_1, e'_2, \dots, e'_n\}$ for the codomain. Recall that the matrix representation of T is given by $[T]^{B'}_B$, where $[T]^{B'}_B$ is defined such that

$$[T]^{B'}_B [v]_B = [T(v)]_{B'}.$$

We can show that

$$[T]^{B'}_B = \Big[[T(e_1)]_{B'} \quad [T(e_2)]_{B'} \quad \dots \quad [T(e_n)]_{B'} \Big].$$

Definition 2.2.8. Given two bases B and B' for V. The **change-of-basis matrix** from B to B' is simply $[I]^{B'}_B$.

Theorem 2.2.9. Given a tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B$, with basis $B_{A\otimes B}$. As shown above, given any pure state $|\psi_{AB}\rangle$, we can find a Schmidt Basis for it, say B_{Sch}. Now denote the CoB matrix from $B_{A\otimes B}$ to B_{Sch} by $[Sch_{AB}]$, i.e.

$$[Sch_{AB}] = [I]^{B_{Sch}}_{B_{A\otimes B}}.$$

Then $[Sch_{AB}]$ is separable.

Proof. Refer to Equations 3 in Theorem 2.2.7. The new (Schmidt) basis for $\mathcal{H}_A \otimes \mathcal{H}_B$ is given by $B_{Sch} = \{|p\rangle_S |q\rangle_S\}$, with $0 \le p \le d_A - 1$, $0 \le q \le d_B - 1$. Writing the old basis vectors in terms of the new basis, we have

$$|j\rangle_A = \sum_p u^\dagger_{pj} |p\rangle_S$$
$$|k\rangle_B = \sum_q v^*_{qk} |q\rangle_S .$$

So

$$|j\rangle_A |k\rangle_B = \sum_p \sum_q u^\dagger_{pj} v^*_{qk} |p\rangle_S |q\rangle_S .$$

From this, we see that $[Sch_{AB}]$ is separable. For concreteness, we write down the details for the simple case where $d_A, d_B = 2$. In this case, we have

$$|0\rangle_A |0\rangle_B = \sum_p \sum_q u^\dagger_{p0} v^*_{q0} |p\rangle_S |q\rangle_S$$
$$|0\rangle_A |1\rangle_B = \sum_p \sum_q u^\dagger_{p0} v^*_{q1} |p\rangle_S |q\rangle_S$$
$$|1\rangle_A |0\rangle_B = \sum_p \sum_q u^\dagger_{p1} v^*_{q0} |p\rangle_S |q\rangle_S$$
$$|1\rangle_A |1\rangle_B = \sum_p \sum_q u^\dagger_{p1} v^*_{q1} |p\rangle_S |q\rangle_S ,$$

so

$$[Sch_{AB}] = \begin{bmatrix}[|00\rangle_{AB}]_{B_{Sch}} & [|01\rangle_{AB}]_{B_{Sch}} & [|10\rangle_{AB}]_{B_{Sch}} & [|11\rangle_{AB}]_{B_{Sch}}\end{bmatrix}$$
$$= \begin{bmatrix} u^\dagger_{00}v^*_{00} & u^\dagger_{00}v^*_{01} & u^\dagger_{01}v^*_{00} & u^\dagger_{01}v^*_{01} \\ u^\dagger_{00}v^*_{10} & u^\dagger_{00}v^*_{11} & u^\dagger_{01}v^*_{10} & u^\dagger_{01}v^*_{11} \\ u^\dagger_{10}v^*_{00} & u^\dagger_{10}v^*_{01} & u^\dagger_{11}v^*_{00} & u^\dagger_{11}v^*_{01} \\ u^\dagger_{10}v^*_{10} & u^\dagger_{10}v^*_{11} & u^\dagger_{11}v^*_{10} & u^\dagger_{11}v^*_{11} \end{bmatrix}$$
$$= (U^T)^* \otimes V^*.$$

□

2.3 Permutation Matrices

Definition 2.3.1. A **permutation matrix** P is a square matrix that has exactly one entry of 1 in each row and each column, and 0's elsewhere. Multiplication of P with another suitably sized matrix, say A, results in permuting the rows (when pre-multiplying, i.e. PA) or columns (when post-multiplying, i.e. AP) of A. Every permutation matrix can be obtained by permuting the rows/columns of the identity matrix I.

2.3.1 Notation

For the purposes of this section, we denote the standard row/column vectors by $\{e_i\}$. Whether e_i assumes row/column form will be clear from the context. It is perhaps best to develop the content here with a concrete example.

Denote an $n \times n$ matrix by $(1, 2, \ldots, n)$, where the integer i represents the ith row of the matrix. Suppose we would like to permute a matrix's rows such that the permuted matrix's rows, arranged in order, are $(\pi(1), \pi(2), \ldots, \pi(n))$, where $\pi : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$ is a permutation on the set of n elements:

$$\pi = \begin{pmatrix} 1 & 2 & \ldots & n \\ \pi(1) & \pi(2) & \ldots & \pi(n) \end{pmatrix}.$$

The appropriate permutation matrix to use is:

$$P = \begin{bmatrix} e_{\pi(1)} & \rightarrow \\ e_{\pi(2)} & \rightarrow \\ \vdots & \\ e_{\pi(n)} & \rightarrow \end{bmatrix},$$

where the e_i's assume row form. Pre-multiplying the matrix by P gives our desired result.

If is it the columns that we wish to permute instead, then let the e_i's assume column form. The appropriate permutation matrix is:

$$P = \begin{bmatrix} e_{\pi(1)} & e_{\pi(2)} & \dots & e_{\pi(n)} \\ \downarrow & \downarrow & \dots & \downarrow \end{bmatrix}.$$

Post-multiplying the matrix by P gives our desired result.

Example 2.3.2. Let $n = 5$, $\pi = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{smallmatrix}\right)$, and

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}.$$

1. (Row Swap)
   Set

$$P_\pi = \begin{bmatrix} e_1 & \rightarrow \\ e_4 & \rightarrow \\ e_2 & \rightarrow \\ e_5 & \rightarrow \\ e_3 & \rightarrow \end{bmatrix}, \text{ such that } P_\pi A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix} \begin{matrix} \mathbf{1} \\ \mathbf{4} \\ \mathbf{2} \\ \mathbf{5} \\ \mathbf{3} \end{matrix}.$$

2. (Column Swap)
   Set

$$P_\pi = \begin{bmatrix} e_1 & e_4 & e_2 & e_5 & e_3 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix}, \text{ such that } AP_\pi = \begin{array}{c} \begin{matrix} \mathbf{1} & \mathbf{4} & \mathbf{2} & \mathbf{5} & \mathbf{3} \end{matrix} \\ \begin{bmatrix} a_{11} & a_{14} & a_{12} & a_{15} & a_{13} \\ a_{21} & a_{24} & a_{22} & a_{25} & a_{23} \\ a_{31} & a_{34} & a_{32} & a_{35} & a_{33} \\ a_{41} & a_{44} & a_{42} & a_{45} & a_{43} \\ a_{51} & a_{54} & a_{52} & a_{55} & a_{53} \end{bmatrix} \end{array}.$$

2.3.2 Properties

1. The set of all $n \times n$-sized permutation matrices P_π forms a group, where the group operation is matrix multiplication and the identity element is the identity matrix. In our description of P_π above, P_π 'behaves' like the permutation π, with the permuted elements being row vectors. Indeed, in group theory jargon, the set of permutation matrices is isomorphic to the symmetric group:

$$\{P_\pi\} \cong S_n$$
$$P_\pi \longleftrightarrow \pi.$$

2. $\{P_\pi\}$ is generated by the set of elementary row-swapping matrices (whose counterparts in S_n are the transpositions).

3. There are $n!$ elements in $\{P_\pi\}$.

Remark 2.3.3. Note that the size of P_π need not be the same as that of A. Suppose A is of size $m \times n$. To permute its rows, P_π is to be of size $m \times m$, and to permute its columns, P_π is to be of size $n \times n$.

3 Main Results

3.1 The Problem in Detail

As briefly discussed in the introduction, we have seen how a particular pair of entangled ququarts, $|\psi\rangle = \frac{1}{2}(|00\rangle + |11\rangle + |22\rangle + |33\rangle)$ could be replicated by two pairs of entangled qubits. $\frac{1}{2}(|00\rangle + |11\rangle + |22\rangle + |33\rangle)$ is thus said to be 'decomposable'. We begin this section by asking: when is a given general $|\psi\rangle$ in a bipartite ququart system decomposable? In a diagram,

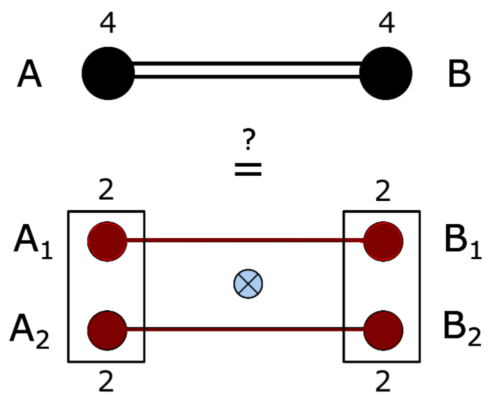


Figure 2: Simulating a pair of entangled ququarts, where $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.

Of course, let us first precisely define what it means for a state to be decomposable.

Definition 3.1.1. A pure state $|\psi\rangle \in (\mathbb{C}^4) \otimes (\mathbb{C}^4)$ is **decomposable** if there exist bipartite states $|\varphi\rangle_{A_1B_1}, |\varphi'\rangle_{A_2B_2}$ of dimensions $4 = 2^2$, and an encoding such that

$$|\psi\rangle = |\varphi\rangle_{A_1B_1} \otimes |\varphi'\rangle_{A_2B_2}.$$

If $|\psi\rangle$ is *not* decomposable, then we say it is **genuinely multilevel entangled**.

In fact, let us not restrict ourselves to bipartite ququart systems. An analogous definition could be made for general bipartite qudit[1] systems as well.

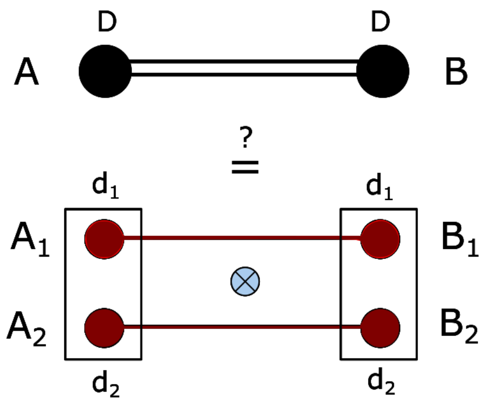


Figure 3: Simulating a pair of entangled qudits, where $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.

Definition 3.1.2. A pure state $|\psi\rangle \in (\mathbb{C}^D) \otimes (\mathbb{C}^D)$ is **decomposable** if there exist bipartite states $|\varphi\rangle_{A_1B_1}, |\varphi'\rangle_{A_2B_2}$ of dimensions d_1^2, d_2^2, where $d_1 \times d_2 = D$ and an encoding such that

$$|\psi\rangle = |\varphi\rangle_{A_1B_1} \otimes |\varphi'\rangle_{A_2B_2}.$$

If $|\psi\rangle$ is *not* decomposable, then we say it is **genuinely multilevel entangled**.

In the next subsection, we reproduce the main ideas of the approach of the authors of [1].

[1] A qudit is a generalization of a qubit to d dimensions. In particular, a qubit is a qudit of dimension 2.

3.1.1 An Existing Approach

First, a general two-ququart state can be written in the Schmidt decomposition as

$$|\psi\rangle = s_0 |00\rangle_{AB} + s_1 |11\rangle_{AB} + s_2 |22\rangle_{AB} + s_3 |33\rangle_{AB} ,$$

where the Schmidt Coefficients are ordered, i.e. $s_0 \geq s_1 \geq s_2 \geq s_3 \geq 0$ (also recall that $\sum_i s_i^2 = 1$). To determine whether $|\psi\rangle$ is decomposable or not, the maximal overlap between $|\psi\rangle$ and all other decomposable states $|\varphi\rangle$ was computed, and was given to be

$$\max_{|\varphi\rangle} | \langle\varphi|\psi\rangle | = \max \text{singval}\,(S)$$

if we choose the encoding between Schmidt Bases, where

$$S = \begin{bmatrix} s_0 & s_1 \\ s_2 & s_3 \end{bmatrix} .$$

It was then claimed (without a direct proof) that $|\psi\rangle$ is decomposable if and only if $\max \text{singval}\,(S) = 1$. In the special case of bipartite ququarts, this is also equivalent to saying that $\det(S) = 0$. For higher dimensional states, it would then be necessary to run through all possible encodings, and it was shown that for a decomposition into $D = d \times d'$, there are

$$N = \frac{(d \times d')!}{\prod_{i=1}^{d} \prod_{j=1}^{d'} (i + j - 1)}$$

permutations (i.e. different ways of encoding) that one has to check.

3.2 An Alternative Method

We propose an 'algebraic' method of detecting decomposability, though we must warn that this is not necessarily much easier to execute than the existing approach. Indeed, as we shall see later, as D increases, the complexity blows up really quickly as well.

For notational simplicity, we shall henceforth use the column representation of a vector, with respect to some specified basis. Let us start with the bipartite ququart system. In its most general form, an arbitrary state $|\psi\rangle$ is written as

$$|\psi\rangle = c_{00}\,|00\rangle_{AB} + \cdots + c_{03}\,|03\rangle_{AB} + c_{10}\,|10\rangle_{AB} + \cdots + c_{33}\,|33\rangle_{AB}$$
$$\xmapsto{B} \begin{bmatrix} c_{00} \\ \vdots \\ c_{33} \end{bmatrix} = [|\psi\rangle]_{B_{AB}},$$

with $B_{AB} = \{|00\rangle_{AB}, |01\rangle_{AB}, \ldots, |33\rangle_{AB}\}$. From Definition 3.1.2 of decomposability, we want to check if this is equal to $|\varphi\rangle_{A_1B_1} \otimes |\varphi'\rangle_{A_2B_2}$ or not, i.e. in column form,

$$[|\psi\rangle]_{B_{AB}} = \begin{bmatrix} c_{00} \\ \vdots \\ c_{33} \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} \otimes \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = [|\varphi\rangle \otimes |\varphi'\rangle]_{B_{A_1B_1A_2B_2}} \tag{5}$$

$$= [|\varphi\rangle]_{B_{A_1B_1}} \otimes [|\varphi'\rangle]_{B_{A_2B_2}} \tag{6}$$

with

$$B_{A_1B_1} = \{|00\rangle_{A_1B_1}, |01\rangle_{A_1B_1}, |10\rangle_{A_1B_1}, |11\rangle_{A_1B_1}\}$$
$$B_{A_2B_2} = \{|00\rangle_{A_2B_2}, |01\rangle_{A_2B_2}, |10\rangle_{A_2B_2}, |11\rangle_{A_2B_2}\}.$$

A priori, this does not quite make sense, for B_{AB} and $B_{A_1B_1A_2B_2}$ are entirely different things. This is where the 'encoding' part comes in - by assigning to each element in B_{AB} an element in $B_{A_1B_1A_2B_2}$. After this encoding, it would make sense to talk about $[|\psi\rangle]_{B_{A_1B_1A_2B_2}}$. Concretely, the only difference between $[|\psi\rangle]_{B_{AB}}$ and $[|\psi\rangle]_{B_{A_1B_1A_2B_2}}$ is that the latter is a permutation of the former (or equivalently, the former is a permutation of the latter).

Example 3.2.1. Consider a toy example. Let $B = \{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$ and $B' = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Consider the state $|\psi\rangle = c_0\,|0\rangle + c_1\,|1\rangle + c_2\,|2\rangle +$

$c_3 |3\rangle$. We have

$$[|\psi\rangle]_B = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

What about $[|\psi\rangle]_{B'}$? That would depend on how we encode B.

1. Encoding: $|0\rangle \mapsto |00\rangle, |1\rangle \mapsto |01\rangle, |2\rangle \mapsto |10\rangle, |3\rangle \mapsto |11\rangle$

$$[|\psi\rangle]_{B'} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

2. Encoding: $|0\rangle \mapsto |00\rangle, |1\rangle \mapsto |10\rangle, |2\rangle \mapsto |11\rangle, |3\rangle \mapsto |01\rangle$

$$[|\psi\rangle]_{B'} = \begin{bmatrix} c_0 \\ c_3 \\ c_1 \\ c_2 \end{bmatrix}$$

3. Encoding: $|0\rangle \mapsto |11\rangle, |1\rangle \mapsto |01\rangle, |2\rangle \mapsto |00\rangle, |3\rangle \mapsto |10\rangle$

$$[|\psi\rangle]_{B'} = \begin{bmatrix} c_2 \\ c_1 \\ c_3 \\ c_0 \end{bmatrix}$$

4. ... and so on, for a total of 24 different encodings.

Remark 3.2.2. Making an encoding with $B_{A_1B_1A_2B_2}$ is equivalent to making an encoding with $B_{A_1A_2B_1B_2}$. More precisely, there is a one-to-one correspondence between the two encodings, since there clearly is an obvious encoding between $B_{A_1B_1A_2B_2}$ and $B_{A_1A_2B_1B_2}$ themselves. To give an example, let us consider our state $|\psi\rangle = \frac{1}{2}(|00\rangle + |11\rangle + |22\rangle + |33\rangle)$ in the Introduction. There, we made the encoding $|0\rangle_A \mapsto |00\rangle_{A_1A_2}, |1\rangle_A \mapsto |01\rangle_{A_1A_2}, |2\rangle_A \mapsto |10\rangle_{A_1A_2}, |3\rangle_A \mapsto |11\rangle_{A_1A_2}$ and similarly for B. This gives us an encoding between B_{AB} and $B_{A_1A_2B_1B_2}$ as follows:

$$\begin{array}{lll}
|00\rangle_{AB} \mapsto |0000\rangle_{A_1A_2B_1B_2} & |01\rangle_{AB} \mapsto |0001\rangle_{A_1A_2B_1B_2} & |02\rangle_{AB} \mapsto |0010\rangle_{A_1A_2B_1B_2} \\
|03\rangle_{AB} \mapsto |0011\rangle_{A_1A_2B_1B_2} & & \\
|10\rangle_{AB} \mapsto |0100\rangle_{A_1A_2B_1B_2} & |11\rangle_{AB} \mapsto |0101\rangle_{A_1A_2B_1B_2} & |12\rangle_{AB} \mapsto |0110\rangle_{A_1A_2B_1B_2} \\
|13\rangle_{AB} \mapsto |0111\rangle_{A_1A_2B_1B_2} & & \\
|20\rangle_{AB} \mapsto |1000\rangle_{A_1A_2B_1B_2} & |21\rangle_{AB} \mapsto |1001\rangle_{A_1A_2B_1B_2} & |22\rangle_{AB} \mapsto |1010\rangle_{A_1A_2B_1B_2} \\
|23\rangle_{AB} \mapsto |1011\rangle_{A_1A_2B_1B_2} & & \\
|30\rangle_{AB} \mapsto |1100\rangle_{A_1A_2B_1B_2} & |31\rangle_{AB} \mapsto |1101\rangle_{A_1A_2B_1B_2} & |32\rangle_{AB} \mapsto |1110\rangle_{A_1A_2B_1B_2} \\
|33\rangle_{AB} \mapsto |1111\rangle_{A_1A_2B_1B_2} & &
\end{array},$$

or equivalently, an encoding between B_{AB} and $B_{A_1B_1A_2B_2}$ as follows:

$$\begin{array}{lll}
|00\rangle_{AB} \mapsto |0000\rangle_{A_1B_1A_2B_2} & |01\rangle_{AB} \mapsto |0001\rangle_{A_1B_1A_2B_2} & |02\rangle_{AB} \mapsto |0100\rangle_{A_1B_1A_2B_2} \\
|03\rangle_{AB} \mapsto |0101\rangle_{A_1B_1A_2B_2} & & \\
|10\rangle_{AB} \mapsto |0010\rangle_{A_1B_1A_2B_2} & |11\rangle_{AB} \mapsto |0011\rangle_{A_1B_1A_2B_2} & |12\rangle_{AB} \mapsto |0110\rangle_{A_1B_1A_2B_2} \\
|13\rangle_{AB} \mapsto |0111\rangle_{A_1B_1A_2B_2} & & \\
|20\rangle_{AB} \mapsto |1000\rangle_{A_1B_1A_2B_2} & |21\rangle_{AB} \mapsto |1001\rangle_{A_1B_1A_2B_2} & |22\rangle_{AB} \mapsto |1100\rangle_{A_1B_1A_2B_2} \\
|23\rangle_{AB} \mapsto |1101\rangle_{A_1B_1A_2B_2} & & \\
|30\rangle_{AB} \mapsto |1010\rangle_{A_1B_1A_2B_2} & |31\rangle_{AB} \mapsto |1011\rangle_{A_1B_1A_2B_2} & |32\rangle_{AB} \mapsto |1110\rangle_{A_1B_1A_2B_2} \\
|33\rangle_{AB} \mapsto |1111\rangle_{A_1B_1A_2B_2} & &
\end{array}.$$

As is becoming painfully apparent now, the complexity of all these encodings can get really tiring. However, this is where the Schmidt Decomposition comes in. Suppose $[|\psi\rangle]_{B_{AB}} = [|\varphi\rangle]_{B_{A_1B_1}} \otimes [|\varphi'\rangle]_{B_{A_2B_2}}$ (after an encoding). Applying the Schmidt Change-of-Basis matrix, which is separable (c.f. subsection 2.2.3), we have

$$\begin{aligned}
[Sch_{AB}][|\psi\rangle]_{B_{AB}} &= [(U^T)^* \otimes V^*][|\varphi\rangle]_{B_{A_1B_1}} \otimes [|\varphi'\rangle]_{B_{A_2B_2}} \\
&= (U^T)^*[|\varphi\rangle]_{B_{A_1B_1}} \otimes V^*[|\varphi'\rangle]_{B_{A_2B_2}},
\end{aligned}$$

where $(U^T)^*$ and V^* are also change-of-basis matrices putting $[|\varphi\rangle]_{B_{A_1B_1}}$ and $[|\varphi'\rangle]_{B_{A_2B_2}}$ respectively into Schmidt form. Thus from now on, we shall deal exclusively with Schmidt Bases.

Since decomposability and separability are closely related, let us make a brief digression. In the next subsection, we develop a method to detect separability, which shall be modified and used in detecting decomposability.

3.2.1 A Short Detour: Separability

Many techniques have been developed to detect separability for a given state, see [6], [7]. We shall add one more to the repertoire.

First, recall from subsection 2.1.3 the notion of separability for column vectors: the column vector $\mathbf{v}$ is said separable if there exist column vectors $\mathbf{a}$ and $\mathbf{b}$ such that

$$\mathbf{v} = \mathbf{a} \otimes \mathbf{b} = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

This definition is compatible with the one made for arbitrary vectors (i.e. $|\psi\rangle$ is separable if there exist $|a\rangle, |b\rangle$ such that $|\psi\rangle = |\varphi\rangle \otimes |\varphi'\rangle$).

Now a general vector $|\psi\rangle \in V \otimes W$ can be expressed as

$$|\psi\rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} |i\rangle_V |j\rangle_W ,$$

where $\{|i\rangle_V\}$ and $\{|j\rangle_W\}$ are bases for V and W respectively. As mentioned (2.1.2), it is customary to assume the ordered basis $\{|11\rangle, \ldots, |1n\rangle, \ldots, |m1\rangle,$

$\ldots, |mn\rangle\}$ for $V \otimes W$. Thus we have

$$[|\psi\rangle]_{B_{V\otimes W}} = \begin{bmatrix} c_{11} \\ \vdots \\ c_{1n} \\ \vdots \\ c_{m1} \\ \vdots \\ c_{mn} \end{bmatrix},$$

which is a matrix of size $mn \times 1$. We now introduce another representative notation: let us define the matrix $M^B_{|\psi\rangle}$ such that $[M^B_{|\psi\rangle}]_{ij} = c_{ij}$, i.e.

$$M^B_{|\psi\rangle} = \begin{bmatrix} c_{11} & \ldots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{m1} & \ldots & c_{mn} \end{bmatrix}.$$

We claim that $|\psi\rangle$ is separable if and only if $M^B_{|\psi\rangle}$ is of rank 1.

Theorem 3.2.3. Let B_V, B_W be bases for the vector spaces V, W (of dimensions m and n respectively), and denote $B_{V\otimes W}$ by B. A vector $|\psi\rangle \in V \otimes W$ is separable if and only if $\text{rank}(M^B_{|\psi\rangle}) = 1$.

Proof.

1. ($\Longrightarrow$) Suppose $|\psi\rangle$ is separable, i.e. $|\psi\rangle = |v\rangle \otimes |w\rangle$ for some $|v\rangle \in V$ and $|w\rangle \in W$. Equivalently in their column representations, we have

$$[|\psi\rangle]_B = \begin{bmatrix} c_{11} \\ \vdots \\ c_{1n} \\ \vdots \\ c_{m1} \\ \vdots \\ c_{mn} \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \otimes \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = [|v\rangle]_{B_V} \otimes [|w\rangle]_{B_W}.$$

Then observe that

$$M^B_{|\psi\rangle} = \begin{bmatrix} c_{11} & \dots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{m1} & \dots & c_{mn} \end{bmatrix} = \begin{bmatrix} v_1 w_1 & \dots & v_1 w_n \\ \vdots & \ddots & \vdots \\ v_m w_1 & \dots & v_m w_n \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} w_1 & \dots & \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} w_n \end{bmatrix},$$

thereby showing that $M^B_{|\psi\rangle}$ is of rank 1.

2. ($\Longleftarrow$) Conversely, suppose $\text{rank}(M^B_{|\psi\rangle}) = 1$. By definition, the dimension of its column space is 1. So picking the first nonzero column, which without loss of generality we let to be the first one, we have

$$M^B_{|\psi\rangle} = \begin{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} 1 & \dots & \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} \beta_n \end{bmatrix}.$$

We can then write

$$[|\psi\rangle]_B = \begin{bmatrix} \alpha_1 1 \\ \alpha_1 \beta_2 \\ \vdots \\ \alpha_1 \beta_n \\ \vdots \\ \alpha_m 1 \\ \alpha_m \beta_2 \\ \vdots \\ \alpha_m \beta_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} \otimes \begin{bmatrix} 1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} = [|v\rangle]_{B_V} \otimes [|w\rangle]_{B_W}.$$

Up to a normalization factor, we have

$$[|v\rangle]_{B_V} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} \quad \text{and} \quad [|w\rangle]_{B_w} = \begin{bmatrix} 1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}.$$

In other words,

$$|\psi\rangle = \left(\sum_{i=1}^{m} \alpha_i |i\rangle\right) \otimes \left(\sum_{j=1}^{n} \beta_j |j\rangle\right).$$

□

Of course, in the theorem above we made use of some particular basis (namely, B_V, B_W and $B = B_{V\otimes W}$). We have to show this technique is independent of the bases B_V, B_W we choose. To do so, consider the new bases B'_V, B'_W and $B' = B'_{V\otimes W}$.

Proposition 3.2.4. $\operatorname{rank}(M^B_{|\psi\rangle}) = 1 \iff \operatorname{rank}(M^{B'}_{|\psi\rangle}) = 1$.

Proof. Let U and V denote the change-of-basis matrices from B_V to B'_V, and from B_W to B'_W. We have

$$\begin{aligned}
\operatorname{rank}(M^B_{|\psi\rangle}) = 1 &\iff M^B_{|\psi\rangle} = \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_m\end{bmatrix} \begin{bmatrix}\beta_1 & \dots & \beta_n\end{bmatrix} \\
&\iff [|\psi\rangle]_B = \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_m\end{bmatrix} \otimes \begin{bmatrix}\beta_1\\ \vdots \\ \beta_n\end{bmatrix} \\
&\iff [|\psi\rangle]_{B'} = U \otimes V [|\psi\rangle]_B = U \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_m\end{bmatrix} \otimes V \begin{bmatrix}\beta_1\\ \vdots \\ \beta_n\end{bmatrix} \\
&\iff M^{B'}_{|\psi\rangle} = U \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_m\end{bmatrix} \begin{bmatrix}\beta_1 & \dots & \beta_n\end{bmatrix} V^T \\
&\iff \operatorname{rank}(M^{B'}_{|\psi\rangle}) = 1,
\end{aligned}$$

where the last equivalence comes from the fact that multiplying a matrix by invertible matrices does not change the matrix's rank. □

Remark 3.2.5. In [6], [7], it was shown that a state is separable if and only if its Schmidt Rank is 1. We can consider that statement to be a *special case* of our theorem here, with B being the Schmidt Basis of the state, since

$$\text{Schmidt Rank} = 1 \iff M^{B}_{|\psi\rangle} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \iff \text{rank}(M^{B}_{|\psi\rangle}) = 1.$$

3.2.2 Decomposability of Bipartite Ququarts

Now, decomposability proper. Again, let us start with the simplest system - a bipartite ququart. For ease of reference, we reproduce the equation we are supposed to tackle (c.f. Eqn 5):

$$\begin{aligned} [|\psi\rangle]_{B_{AB}} = \begin{bmatrix} s_{00} \\ s_{11} \\ s_{22} \\ s_{33} \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} &= [|\varphi\rangle \otimes |\varphi'\rangle]_{B_{A_1B_1A_2B_2}} \\ &= [|\varphi\rangle]_{B_{A_1B_1}} \otimes [|\varphi'\rangle]_{B_{A_2B_2}}, \end{aligned}$$

where we have assumed all the bases to be Schmidt Bases, i.e.

$$\begin{aligned} |\psi\rangle &= s_{00}\,|00\rangle_{AB} + s_{11}\,|11\rangle_{AB} + s_{22}\,|22\rangle_{AB} + s_{33}\,|33\rangle_{AB} \\ |\varphi\rangle &= a\,|00\rangle_{A_1B_1} + b\,|11\rangle_{A_1B_1} \\ |\varphi'\rangle &= c\,|00\rangle_{A_2B_2} + d\,|11\rangle_{A_2B_2}, \end{aligned}$$

and that the Schmidt Coefficients are ordered, i.e. $s_{00} \geq s_{11} \geq s_{22} \geq s_{33} \geq 0$, $a \geq b \geq 0$, $c \geq d \geq 0$.

In accordance with our new 'matrix notation' $M^{B}_{|\psi\rangle}$ developed above, we put this equation into matrix form:

$$M^{B_{AB}}_{|\psi\rangle} = \begin{bmatrix} s_{00} & s_{11} \\ s_{22} & s_{33} \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} c & d \end{bmatrix} = [|\varphi\rangle]_{B_{A_1B_1}} [|\varphi'\rangle]^{T}_{B_{A_2B_2}}.$$

Don't forget the encoding! Here, the encoding takes place between the (Schmidt) Bases

$$B_{AB} = \{|00\rangle_{AB}, |11\rangle_{AB}, |22\rangle_{AB}, |33\rangle_{AB}\}$$
$$B_{A_1B_1A_2B_2} = \{|0000\rangle_{A_1B_1A_2B_2}, |0011\rangle_{A_1B_1A_2B_2}, |1100\rangle_{A_1B_1A_2B_2}, |1111\rangle_{A_1B_1A_2B_2}\},$$

so the term $M_{|\psi\rangle}^{B_{A_1B_1A_2B_2}}$ is well-defined.

Example 3.2.6.

1. Encoding:

$$|00\rangle_{AB} \mapsto |0000\rangle_{A_1B_1A_2B_2}, \quad |11\rangle_{AB} \mapsto |0011\rangle_{A_1B_1A_2B_2}$$
$$|22\rangle_{AB} \mapsto |1100\rangle_{A_1B_1A_2B_2}, \quad |33\rangle_{AB} \mapsto |1111\rangle_{A_1B_1A_2B_2}$$

$$M_{|\psi\rangle}^{B_{A_1B_1A_2B_2}} = \begin{bmatrix} s_{00} & s_{11} \\ s_{22} & s_{33} \end{bmatrix}$$

2. Encoding:

$$|00\rangle_{AB} \mapsto |0000\rangle_{A_1B_1A_2B_2}, \quad |11\rangle_{AB} \mapsto |1100\rangle_{A_1B_1A_2B_2}$$
$$|22\rangle_{AB} \mapsto |0011\rangle_{A_1B_1A_2B_2}, \quad |33\rangle_{AB} \mapsto |1111\rangle_{A_1B_1A_2B_2}$$

$$M_{|\psi\rangle}^{B_{A_1B_1A_2B_2}} = \begin{bmatrix} s_{00} & s_{22} \\ s_{11} & s_{33} \end{bmatrix}$$

3. Encoding:

$$|00\rangle_{AB} \mapsto |1111\rangle_{A_1B_1A_2B_2}, \quad |11\rangle_{AB} \mapsto |0011\rangle_{A_1B_1A_2B_2}$$
$$|22\rangle_{AB} \mapsto |0000\rangle_{A_1B_1A_2B_2}, \quad |33\rangle_{AB} \mapsto |1100\rangle_{A_1B_1A_2B_2}$$

$$M_{|\psi\rangle}^{B_{A_1B_1A_2B_2}} = \begin{bmatrix} s_{22} & s_{11} \\ s_{33} & s_{00} \end{bmatrix}$$

Remark 3.2.7 (Caution!). Note that the last example is not always valid! This is due to our assumption that the Schmidt Coefficients are ordered. Thus if the largest and/or smallest coefficients s_{00}, s_{33} are unique, i.e. there are no duplicates among the other Schmidt Coefficients, and if there were an encoding that leads to decomposability, then $s_{00} = ab$ and $s_{33} = cd$ with no other possible choices. In the next subsection, after dealing with general bipartite systems, we shall provide a lower bound for different encodings leading to decomposability (provided a least one exists). The result makes use of the permutation matrices introduced among the mathematical tools above.

Let us now apply this technique to affirm what we already know to be true: that the maximally entangled state $\frac{1}{2}(|00\rangle_{AB} + |11\rangle_{AB} + |22\rangle_{AB} + |33\rangle_{AB})$ is decomposable.

Example 3.2.8. The maximally entangled state $|\psi\rangle = \frac{1}{2}(|00\rangle_{AB} + |11\rangle_{AB} + |22\rangle_{AB} + |33\rangle_{AB})$ is decomposable. In fact, there are $4! = 24$ different encodings, all of which lead to decomposability. This should be clear, since for every single encoding we have

$$M_{|\psi\rangle}^{B_{A_1B_1A_2B_2}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

which is of rank 1. While in the introduction we chose a specific encoding, this would work for any other as well, due to the Schmidt Coefficients being identical. The encoding used in the introduction corresponds to:

$$\begin{aligned} |00\rangle_{AB} &\longmapsto |0000\rangle_{A_1A_2B_1B_2} & |11\rangle_{AB} &\longmapsto |0101\rangle_{A_1A_2B_1B_2} \\ |22\rangle_{AB} &\longmapsto |1010\rangle_{A_1A_2B_1B_2} & |33\rangle_{AB} &\longmapsto |1111\rangle_{A_1A_2B_1B_2}, \end{aligned}$$

or equivalently

$$\begin{aligned} |00\rangle_{AB} &\longmapsto |0000\rangle_{A_1B_1A_2B_2} & |11\rangle_{AB} &\longmapsto |0011\rangle_{A_1B_1A_2B_2} \\ |22\rangle_{AB} &\longmapsto |1100\rangle_{A_1B_1A_2B_2} & |33\rangle_{AB} &\longmapsto |1111\rangle_{A_1B_1A_2B_2}. \end{aligned}$$

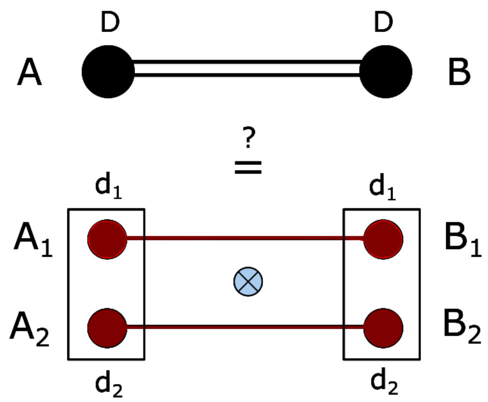
Example 3.2.9. The state $|\psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle_{AB} + |11\rangle_{AB} + |22\rangle_{AB})$ is not decomposable. For any encoding, we have

$$M_{|\psi\rangle}^{B_{A_1B_1A_2B_2}} = \frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} \text{ or } \frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} \text{ or } \frac{1}{\sqrt{3}}\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix} \text{ or } \frac{1}{\sqrt{3}}\begin{bmatrix}0 & 1\\1 & 1\end{bmatrix},$$

none of which are of rank 1.

3.2.3 Decomposability of General Bipartite Systems

The decomposability of general $D \times D$ systems is assessed in an identical manner to what has been done for the bipartite ququarts, with the only change being dimensionality.



Example 3.2.10.

1. $D = 6,\ d_1 = 2, d_2 = 3;$
   $|\psi\rangle = \frac{1}{\sqrt{6}}(|00\rangle_{AB} + |11\rangle_{AB} + |22\rangle_{AB} + |33\rangle_{AB} + |44\rangle_{AB} + |55\rangle_{AB})$

$$M_{|\psi\rangle}^{B_{A_1B_1A_2B_2}} = \frac{1}{\sqrt{6}}\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\end{bmatrix}$$

   for any encoding, so $|\psi\rangle$ is decomposable.

2. $D = 6,\ d_1 = 2, d_2 = 3;$
   Making the encoding

$$\begin{aligned} |00\rangle_{AB} &\longmapsto |0000\rangle_{A_1B_1A_2B_2} & |11\rangle_{AB} &\longmapsto |0011\rangle_{A_1B_1A_2B_2} \\ |22\rangle_{AB} &\longmapsto |1100\rangle_{A_1B_1A_2B_2} & |33\rangle_{AB} &\longmapsto |1111\rangle_{A_1B_1A_2B_2} \\ |44\rangle_{AB} &\longmapsto |0022\rangle_{A_1B_1A_2B_2} & |55\rangle_{AB} &\longmapsto |1122\rangle_{A_1B_1A_2B_2} \end{aligned}$$

   gives

$$M_{|\psi\rangle}^{B_{A_1B_1A_2B_2}} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

   so $|\psi\rangle$ is decomposable.

3. $D = 12,\ d_1 = 3, d_2 = 4;$
   $|\psi\rangle = s_{00}|00\rangle_{AB} + s_{11}|11\rangle_{AB} + s_{22}|22\rangle_{AB} + s_{33}|33\rangle_{AB} + s_{44}|44\rangle_{AB} + s_{55}|55\rangle_{AB} + s_{66}|66\rangle_{AB} + s_{77}|77\rangle_{AB} + s_{88}|88\rangle_{AB} + s_{99}|99\rangle_{AB} + s_{1010}|1010\rangle_{AB} + s_{1111}|1111\rangle_{AB}$

   There are 12! different encodings to check before we can determine whether $|\psi\rangle$ is decomposable. Is there any way to reduce this complexity?

From the last example above, we see how the number of possible encodings blows up factorially. It might very well be that $|\psi\rangle$ is decomposable, and that the encodings allowing this are obvious, but often it is not easy to tell. To establish that $|\psi\rangle$ is *not* decomposable, we have to run through all the encodings and make sure all the resulting $M_{|\psi\rangle}^{B_{A_1B_1A_2B_2}}$'s are not of rank 1.

Proposition 3.2.11.

1. The *maximum* number of encodings we have to run through to decide decomposability for $D = d_1 \times d_2$ is given by

$$\frac{(d_1 d_2)!}{d_1! d_2!}.$$

2. If an encoding allowing decomposability exists, there exist *at least* $(d_1 - 2)!(d_2 - 2)!$ other such encodings.

Proof.

1. For each encoding, we have to find out the resulting $M_{|\psi\rangle}^{B_{A_1 B_1 A_2 B_2}}$'s rank. Are there different encodings resulting in the same rank for $M_{|\psi\rangle}^{B_{A_1 B_1 A_2 B_2}}$? Practically, are there permutations of a matrix's elements such that the permuted matrix has the same rank as before?

   Yes! It is an elementary result from linear algebra that row and column swaps do the job. The question thus boils down to how many different results we can obtain by applying row swaps/column swaps/combinations of both on the matrix. This is where the permutation matrices in Section 2.3 comes in handy. Pre-multiplying a matrix by a permutation matrix permutes the matrix's rows, while post-multiplying permutes the columns. Also, from the properties 2.3.2, we deduce that there are at most $d_1!d_2!$ different matrices that can result from these operations, all of which have the same rank. Thus the maximum number of encodings we have to run through is divided by this number, giving

$$\frac{(d_1 d_2)!}{d_1!d_2!}.$$

2. The reasoning here is the same as above, but we have to be a little cautious. If the largest and smallest Schmidt Coefficients are unique, then we cannot permute them out, c.f Remark 3.2.7. Thus the number of rows/columns we can permute are each reduced by two. Now if we have an encoding that leads to decomposability, then permuting the 'inner' rows/columns would lead to the same rank (= 1) as well, hence decomposability too.

□

Remark 3.2.12. Note that the proposition above works even for the worst-case combinatorial scenario, when all the Schmidt Coefficients are unique. If there are identical elements, the complexity reduces drastically. An extreme example would be the state $|\psi\rangle = \frac{1}{2}(|00\rangle_{AB} + |11\rangle_{AB} + |22\rangle_{AB} + |33\rangle_{AB})$, where all encodings lead to decomposability.

3.3 Extensions of The Method

Now we ask - are the techniques we have developed so far applicable to more complex systems as well? We shall consider two different extensions of the bipartite system, namely multipartite systems and 'finer' decompositions.

3.3.1 General n-partite Systems

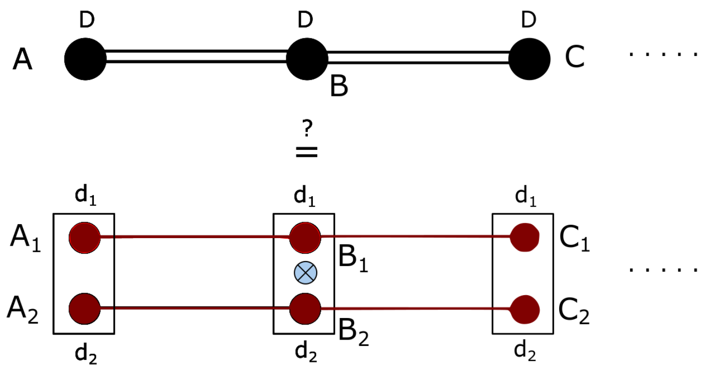


Figure 4: Simulating n entangled qudits

Analogously to Definition 3.1.2, we have

Definition 3.3.1. A pure state $|\psi\rangle \in (\mathbb{C}^D) \otimes (\mathbb{C}^D) \otimes (\mathbb{C}^D)$ is **decomposable** if there exist tripartite states $|\varphi\rangle_{A_1B_1C_1}, |\varphi'\rangle_{A_2B_2C_2}$ of dimensions d_1^3, d_2^3, where $d_1 \times d_2 = D$ and an encoding such that

$$|\psi\rangle = |\varphi\rangle_{A_1B_1C_1} \otimes |\varphi'\rangle_{A_2B_2C_2}.$$

The main result is not applicable here as for n-partite systems where $n > 2$, there is no Schmidt Decomposition (c.f. [4], [5]). There is a weaker, generalized form of the Schmidt Decomposition, which we shall include here for completeness. The author however is unable to make good use of it in this problem.

Theorem 3.3.2 (Generalized Schmidt Decomposition). [5] Suppose we have a pure state $|\Psi\rangle \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$, where $n \geq 3$ and $\dim(\mathcal{H}_1) = \dim(\mathcal{H}_2) = \cdots = \dim(\mathcal{H}_n) = d \geq 2$. Then for $r = 1, 2, \ldots, n$ there is a basis $\{|\psi_i^r\rangle : i = 1, \ldots, d\}$ of $\mathcal{H}_r$ such that in the expansion

$$|\Psi\rangle = \sum_{i_1 \ldots i_n} c_{i_1 \ldots i_n} |\psi_{i_1}^1\rangle |\psi_{i_2}^2\rangle \ldots |\psi_{i_n}^n\rangle$$

the coefficients $c_{i_1 \ldots i_n}$ have the following properties:

1. $c_{jii \ldots i} = c_{iji \ldots i} = \cdots = c_{ii \ldots ij} = 0$ if $1 \leq i < j \leq d$
2. $c_{i_1 \ldots i_n}$ is real and non-negative if at most one of the i_r differs from d
3. $|c_{ii \ldots i}| \geq |c_{j_1 \ldots j_n}|$ if $i \leq j_r$, $r = 1, \ldots, n$.

Remark 3.3.3. For the special case of the tripartite qubit, i.e. $n = 3$, $d = 2$, we have (c.f. [4])

$$|\Psi\rangle = \lambda_0 |000\rangle + \lambda_1 e^{i\varphi} |100\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle + \lambda_4 |111\rangle$$

where $\lambda_i \geq 0$, $0 \leq \varphi \leq \pi$ and $\sum_i \lambda_i^2 = 1$.

3.3.2 Finer Decompositions

For simplicity, let us consider a three-component decomposition (all decompositions so far were two-component decomposition). The general definition of the n-component decomposition is easily formulated.

Definition 3.3.4. A pure state $|\psi\rangle \in (\mathbb{C}^D) \otimes (\mathbb{C}^D)$ is **3-decomposable** if there exist bipartite states $|\varphi\rangle_{A_1B_1}, |\varphi'\rangle_{A_2B_2}, |\varphi''\rangle_{A_3B_3}$ of dimensions d_1^2, d_2^2, d_3^2, where $d_1 \times d_2 \times d_3 = D$ and an encoding such that

$$|\psi\rangle = |\varphi\rangle_{A_1B_1} \otimes |\varphi'\rangle_{A_2B_2} \otimes |\varphi''\rangle_{A_3B_3}.$$

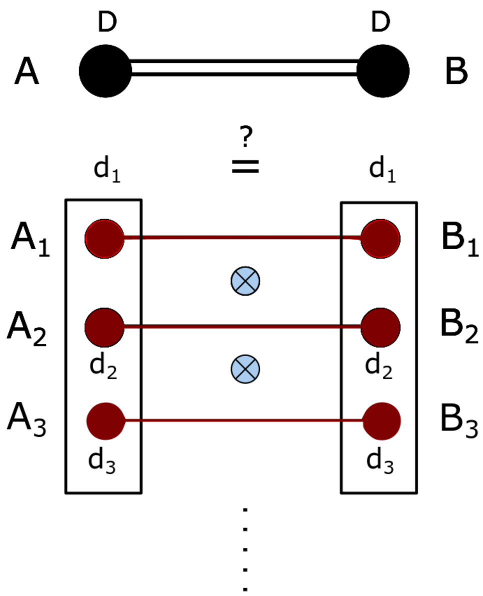


Figure 5: A finer simulation of entangled qudits

In this case, the methods developed above is applicable, simply via **iteration** - if AB decomposes into A_1B_1 and $A'B'$, then we attempt to decompose $A'B'$ itself. This is best demonstrated via a concrete example, using by now what should be our favourite state - the maximally entangled state.

Example 3.3.5. Let $D = 8$, and $d_1 = d_2 = d_3 = 2$. The state $|\psi\rangle = \frac{1}{2\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB} + |22\rangle_{AB} + |33\rangle_{AB} + |44\rangle_{AB} + |55\rangle_{AB} + |66\rangle_{AB} + |77\rangle_{AB})$ is 3-decomposable.

Proof. To see this, first make the encoding between

$$B_{AB} = \{|00\rangle_{AB}, |11\rangle_{AB}, |22\rangle_{AB}, |33\rangle_{AB}, |44\rangle_{AB}, |55\rangle_{AB}, |66\rangle_{AB}, |77\rangle_{AB}, \}$$

$$\begin{aligned} B_{(A_1B_1)(A'B')} = \{&|00\rangle_{A_1B_1}|00\rangle_{A'B'}, |00\rangle_{A_1B_1}|11\rangle_{A'B'}, \\ &|00\rangle_{A_1B_1}|22\rangle_{A'B'}, |00\rangle_{A_1B_1}|33\rangle_{A'B'}, \\ &|11\rangle_{A_1B_1}|00\rangle_{A'B'}, |11\rangle_{A_1B_1}|11\rangle_{A'B'}, \\ &|11\rangle_{A_1B_1}|22\rangle_{A'B'}, |11\rangle_{A_1B_1}|33\rangle_{A'B'}\}. \end{aligned}$$

Doing so gives

$$M_{|\psi\rangle}^{B_{(A_1B_1)(A'B')}} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

so $|\psi\rangle$ is (2-)decomposable, into

$$|\varphi\rangle = \frac{1}{\sqrt{2}}(|00\rangle_{A_1B_1} + |11\rangle_{A_1B_1})$$

and

$$|\alpha\rangle = \frac{1}{2}(|00\rangle_{A'B'} + |11\rangle_{A'B'} + |22\rangle_{A'B'} + |33\rangle_{A'B'}).$$

Now we further attempt to decompose $|\alpha\rangle$. If this fails, $|\psi\rangle$ is still (2-)decomposable. If this works, $|\psi\rangle$ is then 3-decomposable. Make the encoding between

$$\begin{aligned} B_{A'B'} &= \{|00\rangle_{A'B'}, |11\rangle_{A'B'}, |22\rangle_{A'B'}, |33\rangle_{A'B'}\} \\ B_{A_2B_2A_3B_3} &= \{|00\rangle_{A_2B_2}|00\rangle_{A_3B_3}, |00\rangle_{A_2B_2}|11\rangle_{A_3B_3}, \\ &\qquad |11\rangle_{A_2B_2}|00\rangle_{A_3B_3}, |11\rangle_{A_2B_2}|11\rangle_{A_3B_3}\}, \end{aligned}$$

we have

$$M_{|\alpha\rangle}^{B_{A_2B_2A_3B_3}} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

which is certainly of rank 1. So $|\alpha\rangle$ itself is decomposable into

$$|\varphi'\rangle = \frac{1}{\sqrt{2}}(|00\rangle_{A_2B_2} + |11\rangle_{A_2B_2})$$

and

$$|\varphi''\rangle = \frac{1}{\sqrt{2}}(|00\rangle_{A_3B_3} + |11\rangle_{A_3B_3}).$$

Thus, $|\psi\rangle$ is 3-decomposable, where

$$|\psi\rangle_{AB} = |\varphi\rangle_{A_1B_1} \otimes |\alpha\rangle_{A'B'} = |\varphi\rangle_{A_1B_1} \otimes |\varphi'\rangle_{A_2B_2} \otimes |\varphi''\rangle_{A_3B_3}.$$

□

4 Python Code for evaluating Bipartite Systems

```
#Takes in matrix (list of lists). This function checks
    for L.I. of rows, or not.
def matrix_rank_is_1(mat):
    if find_non_zero_in_matrix(mat) == 'Bad Matrix':
        return False
    else:
        i,j = find_non_zero_in_matrix(mat)
        ref_row = mat[i]
        for n in range(0,len(mat)):
            if n != i:
                row1 = ref_row.copy()
                row2 = (mat[n]).copy()
                row1_ind = row1[j]
                row2_ind = row2[j]
                if tuple(map(lambda x: x*row2_ind, row1)
    ) != tuple(map(lambda x: x*row1_ind, row2)):
                    return False
        return True

#Auxiliary
def find_non_zero_in_matrix(mat):
    for i in range(0,len(mat)):
        for j in range(0,len(mat[0])):
            if mat[i][j] != 0:
                return (i,j)
    return 'Bad Matrix'
```

```
#Takes in list, returns a list containing all permuted
  versions of the input list, including duplicates.
def perm_list(lst):
    if len(lst) == 1:
        return [lst]
    result = []
    for i in range(0,len(lst)):
        temp = lst.copy()
        temp.pop(i)
        result.extend(list(map(lambda x: [lst[i]]+x,
  perm_list(temp))))
    return result

#Used to count number of different permutations. Returns
   a dictionary where the keys represent the permuted
  matrices and the values the number of duplicates.
def count(iterable):
    result = {}
    for item in iterable:
        if item not in result.keys():
            result[item] = 1
        else:
            result[item] += 1
    return result

#Converts matrix to list.
def flat(mat):
    result = []
    for row in mat:
```

```
        result.extend(row)
    return result

#Converts list to mxn matrix.
def conv_to_mn(lst,m,n):
    result = []
    counter = 0
    for i in range(0,m):
        result.append(lst[counter:counter+n])
        counter += n
    return result

#Auxiliary
def accum(iterable,operator,init):
    res = init
    for i in iterable:
        res = operator(res,i)
    return res

#Auxiliary
def factorial(n):
    return accum(range(n+1)[1:],lambda x,y: x*y, 1)

#Auxiliary
def num_perm(mat):
    return accum(map(lambda x: factorial(x), count(flat(
    mat)).values()),lambda x,y: x*y, 1)

#Takes in a matrix, returns all permuted matrices of
    rank one together with the number of different
```

```
    encodings (if they exist), otherwise returns 'None'.
def permutations_of_rank1(mat):
    result = []
    m = len(mat)
    n = len(mat[0])
    first_entry = mat[0][0]
    last_entry = mat[m-1][n-1]
    num_perms = num_perm(mat)
    exists = False
    permutations = set(map(lambda x : tuple(x),
  perm_list(flat(mat)[1:m*n-1])))
    for p in permutations:
        curr_mat = conv_to_mn([first_entry]+list(p)+[
  last_entry],m,n)
        if matrix_rank_is_1(curr_mat):
            exists = True
            result.append((curr_mat,num_perms))
    if exists:
        for i in result:
            print(i)
    else:
        print('None')

#Here are some sample runs
>>> matbell2 = [[1,1],[1,1]]
>>> permutations_of_rank1(matbell2)
([[1, 1], [1, 1]], 24)

>>> matbell3 = [[1,1,1],[1,1,1],[1,1,1]]
>>> permutations_of_rank1(matbell3)
```

```
([[1, 1, 1], [1, 1, 1], [1, 1, 1]], 362880)

>>> mat = [[1,1],[0,0]]
>>> permutations_of_rank1(mat)
([[1, 0], [1, 0]], 4)
([[1, 1], [0, 0]], 4)

>>> mat = [[1,1],[1,0]]
>>> permutations_of_rank1(mat)
None

>>> mat = [[8,5,3],[2,1,0]]
>>> permutations_of_rank1(mat)
None

>>> mat = [[8,4,4],[4,2,2],[2,1,1]]
>>> permutations_of_rank1(mat)
([[8, 4, 4], [4, 2, 2], [2, 1, 1]], 72)
([[8, 4, 2], [4, 2, 1], [4, 2, 1]], 72)

>>> mat = [[8,6,4,2],[4,3,2,1],[0,0,0,0]]
>>> permutations_of_rank1(mat)
([[8, 6, 2, 4], [4, 3, 1, 2], [0, 0, 0, 0]], 96)
([[8, 4, 6, 2], [4, 2, 3, 1], [0, 0, 0, 0]], 96)
([[8, 6, 4, 2], [4, 3, 2, 1], [0, 0, 0, 0]], 96)
([[8, 2, 4, 6], [4, 1, 2, 3], [0, 0, 0, 0]], 96)
([[8, 2, 6, 4], [4, 1, 3, 2], [0, 0, 0, 0]], 96)
([[8, 4, 2, 6], [4, 2, 1, 3], [0, 0, 0, 0]], 96)

>>> mat = [[8,6,4,2],[4,3,2,1],[1,0,0,0]]
```

```
>>> permutations_of_rank1(mat)
None
```

5 Conclusion

Thus, we finish the report. We have devised a technique to tell whether a general bipartite pure state is decomposable or not, and made attempts to extend this method to n-partite systems (didn't work) and 'finer' decompostions (worked). In their paper [1], the authors came up with a definition of decomposability for mixed states, but no progress was made on it. Future work could be done on this, and perhaps, similarly to 'measures' of entanglement, we could come up with 'measures' of genuine multilevel entanglement as well?

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