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Information and Neutrino Physics

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Abstract

Using information theoretic arguments to constraint the linear Dirac equation, we derive some generalized Dirac equation. Unlike in [11], we relax the locality constraint. Despite the removal of locality, the generalized Dirac equations are found to be nonlinear and Lorentz violating.

Modified dispersion relations are obtained from the generalized Dirac equation, we then apply these to neutrino oscillations. Comparing with the conventional theory, the modified Dirac equations have various advantages such as sensitivity to individual masses of neutrinos and the removal of need for neutrinos to have masses.

In summary, we are able to generate a class of non-local, nonlinear Dirac equation and it can be used as a probe for quantum linearity in future experiments.

Chapter 1

Introduction

Evolution equations used in quantum theories are linear. They have led to results that agree well with findings from experiments and no deviations from quantum linearity have been found till the present [3] [9]. As a result, any deviation from linearity has to be small and below the accuracy of those experiments. However many nonlinear quantum theories have been proposed and many of them have been useful in describing various physical phenomena in areas like optics, condensed matter physics, particle physics, atomic physics and nuclear physics [1] [15]. These nonlinear equations that describe the various phenomena serve as effective equations.

As mentioned in Ref. [14], a viewpoint is that quantum linearity may be related to space-time symmetry and hence deviation in quantum linearity will mean a corresponding violation of the space-time symmetry, namely Lorentz violation. At present, there is no experimental evidence of Lorentz violation and hence any such violation must be small like that of quantum nonlinearity mentioned earlier.

Hence, we will look for such violations at high energies or at very short distances. The Dirac equation is a relativistic wave equation that is consistent with both quantum mechanics and special relativity. In this thesis, one of the main objectives is to find a generalized Dirac equation. The approach to doing so will be by applying information theoretic arguments to the Dirac equation. Information theoretic approach is also known as the maximum entropy principle. It is a method to infer probability distributions that gives the least biased description of the state of system [14]. It has been used to infer probability distributions in statistical mechanics [6] [7] and also the Schrödinger equation in non-relativistic quantum mechanics [16].

Using information theoretic arguments, we wish to construct a Lagrangian of the form

$$\boldsymbol{\mathcal{L}} = \bar{\boldsymbol{\psi}}(\boldsymbol{i\gamma^{\mu}\partial_{\mu}} - \boldsymbol{m})\boldsymbol{\psi} + \boldsymbol{F}\,, \qquad \textbf{(1. 1)}$$

where F is an information measure term that will be minimized using the Euler-Lagrange equation when deriving the equation of motion as required by information theoretic. It is a function of the wavefunction ψ and its adjoint $\bar{\psi}$.

From here we will then proceed to apply constraints obtained from information theoretic arguments to F to get various possible forms of F. Similarly, we do not demand nonlinearity but will find that it is an unavoidable consequence of the information-theoretic generalization. Unlike in [11], we do not impose locality as a constraint. Lorentz violation is also a consequence of the constraints. These generalised Dirac equations are interpreted as encoding new physics of higher energies. The generalized equation will lead to a modified energy dispersion relation which we will proceed to apply it to neutrinos oscillations. We will probe how the new equation affects the oscillation probabilities and non-linearity in future neutrino oscillations experiments [12]. We also find that some models have the advantage over conventional theory of enabling us to find masses of individual neutrinos.

Finally we will discuss the possible applications of the generalisation. That is the possibility of describing neutrino oscillations without massive neutrinos.

The thesis is outlined as follows: In chapter 2, we will describe information-theoretic approach and the constraints derived from it. In chapter 3 we will discuss plane wave solution and how energy dispersion relation can be obtained from the Dirac equation. In chapter 4 we will discuss neutrino oscillation and apply our generalized Dirac equation to it. In chapter 5, we will discuss the future aims of this study and also some of the possible applications. We will end with a summary at chapter 6.

Chapter 2

Information Theoretic

In this chapter, we will be considering how information theoretic arguments can be used to form generalised Dirac equations. Information-theoretic arguments, also called maximum entropy principle, is a method to infer probability distributions that gives the least biased description of the state of system[1] [14]. As quantum mechanics is also probabilistic in nature, we hope to apply information theoretic arguments to it.

This method has been used to derive the Schrödinger equation which is in the non-relativistic regime [16]. The nonlinear generalization of the Schrödinger equation has interesting properties and applications in areas such as quantum cosmology [13].

Now we ask if it is possible to apply it to the relativistic regime. Instead of probability density and its adjoint, we will be using the wavefunction and its adjoint in our information measure. Starting with the linear Dirac Lagrangian, we will add a information measure term by considering our information theoretic argument constraints. We will then find out that nonlinearity is an unavoidable consequence of our information theoretic generalization.

In the next section we will be looking at the constraints and their physical implications from information theoretic point of view. In Section (2.2) we will show how we construction our information measure from the constraints and that Lorentz

[1] Refer to Ref. [10] for a concise description of information theoretic argument in statistical mechanics.

violation is unavoidable. In Section (2.3) we will discuss the minimization constraint and work through the process of doing it. In Section (2.4) we will look at the 3 different models of the nonlinear, Lorentz violating Dirac equations.

2.1 Constraints

We are interested in information measure, $I = \int F d^4x$. Here our assumption is that ψ and $\bar{\psi}$ in F contracts to form scalars. The information measure should satisfy the following conditions.

- [C1] Homogeneity: The information measure is required to be invariant under the scaling $F(\lambda\psi, \lambda\bar{\psi}) = \lambda^2 F(\psi, \bar{\psi})$ so that the modified equation retain this property and is allowed to be freely normalized.

- [C2] Uncertainty: As the information measure decree the amount of information, we require it to decrease as our wavefunction ψ goes towards a uniform distribution. A way to achieve this will be to include derivatives of ψ. Since the linear Dirac equation already contains this derivative, this seems to be a natural and simple solution.

- [C3] Positivity: The information measure should be non-negative for generic ψ. Thus F should be real and non-negative.

- [C4] Minimisation: The maximum uncertainty principle requires that the information measure is minimum when we extremises the total action to obtain an equation of motion.

Condition [C1] is satisfied by the usual linear Dirac Lagrangian, while conditions [C2], [C3] and [C4] are chosen to incorporate information theoretic principle.

2.2 Construction

As Ref [11] has already looked at information measures that satisfy the locality constraints, we will attempt to look for nonlocal generalizations which also simultaneously satisfy the other constraints. One example of such a term will be of the form $F_1 = Re^{P/Q}$. Here R, P and Q are functions constructed from the wavefunction and its adjoint. In order to satisfy [C1], we require the exponent to be of the form P/Q.

Also R must be real and positive to satisfy [C3]. We can do so by using R^2 instead of R, however this will lead to a violation of [C1]. Hence we will enforce [C3] by using $\psi^\dagger$ instead of $\bar{\psi}$ when contracting with ψ. Doing this will give rise to Lorentz violation and it cannot be avoided due to the constraints. We introduce a background vector field responsible for the Lorentz violation in the form $\bar{\psi}A_\mu\gamma^\mu\psi$, with $A_\mu = (A, 0,0,0)$ in the frame where positivity is enforced. Referring to ref. [11], such terms are not invariant under particle Lorentz transformation. Also, under observer Lorentz transformations [11], only observers who are only rotated with respect to the initial frame can interpret the generalized action in information-theoretic terms. Here, R and Q will contain such a Lorentz violating term to make them positive.

As for [C2], it requires that R contains derivatives of the wavefunction. However as will be seen later, our R in this example does not contain any derivatives of the wavefunction and hence does not satisfy [C2]. It is to be noted that while R does not satisfy [C2], we do require it to be very small and thus [C2] is violated minimally.

Other forms that satisfy our above four constraints and are nonlocal are $F_2 = \frac{P^2}{Q} e^{P^2/Q^2}$ and $F_3 = \frac{P^2}{Q} e^{-Q/P}$ and they follow similar arguments as above.

2.3 Minimization

We will now proceed to look at the final constraint [C4] minimization. Note that the positivity constraint [C3] does not imply that [C4] is satisfied. Looking at the above example $F_2 = \frac{P^2}{Q} e^{P^2/Q^2}$, we consider the variation $\psi \to \psi + \varepsilon\delta\psi$ of a Lagrangian[2] about a solution of the equation of motion. We will denote $X(\psi, \bar{\psi}) \to X(\psi, \bar{\psi}) + \varepsilon X(\delta\psi, \bar{\psi}) \equiv X + \varepsilon X'$ where X refers to P and Q. The real parameter ε keeps track of the order of infinitesimals and the deviation $\delta\psi$ is chosen such that P' and Q' are real. We will now find the second order derivative of the information measure term with respect to ε and show that it is positive definite.

$$F_2 = \frac{P(\psi, \bar{\psi})^2}{Q(\psi, \bar{\psi})} exp \frac{P(\psi, \bar{\psi})^2}{Q(\psi, \bar{\psi})^2} \quad \textbf{(2. 1)}$$

$$\frac{dF_2}{d\varepsilon} = \left[exp \frac{P^2}{Q^2}\right] \left[\frac{2(P + \varepsilon P')P'}{(Q + \varepsilon Q')} - \frac{(P + \varepsilon P')^2 Q'}{(Q + \varepsilon Q')^2} + \frac{(P + \varepsilon P')^2}{(Q + \varepsilon Q')} \left(\frac{2(P + \varepsilon P')P'}{(Q + \varepsilon Q')^2} - \frac{2(P + \varepsilon P')^2 Q'}{(Q + \varepsilon Q')^3}\right)\right] \quad \textbf{(2. 2)}$$

[2] In this variation, $\bar{\psi}$ is treated as a independent variable and kept fixed.

$$\frac{d^2F_2}{d\varepsilon^2} = \left[exp\frac{P^2}{Q^2}\right]\left\{\left(\frac{2(P+\varepsilon P')P'}{(Q+\varepsilon Q')^2} - \frac{2(P+\varepsilon P')^2Q'}{(Q+\varepsilon Q')^3}\right)\left[\frac{2(P+\varepsilon P')P'}{(Q+\varepsilon Q')} - \frac{(P+\varepsilon P')^2Q'}{(Q+\varepsilon Q')^2}\right.\right.$$

$$\left.+\frac{(P+\varepsilon P')^2}{(Q+\varepsilon Q')}\left(\frac{2(P+\varepsilon P')P'}{(Q+\varepsilon Q')^2} - \frac{2(P+\varepsilon P')^2Q'}{(Q+\varepsilon Q')^3}\right)\right] + \frac{2P'^2}{Q+\varepsilon Q'}$$

$$-\frac{4(P+\varepsilon P')P'Q'}{(Q+\varepsilon Q')^2} + \frac{2(P+\varepsilon P')^2Q'^2}{(Q+\varepsilon Q')^3}$$

$$+\left(\frac{2(P+\varepsilon P')P'}{Q+\varepsilon Q'} - \frac{2(P+\varepsilon P')^2Q'}{(Q+\varepsilon Q')^2}\right)\left(\frac{2(P+\varepsilon P')P'}{(Q+\varepsilon Q')^2}\right.$$

$$\left.-\frac{2(P+\varepsilon P')^2Q'}{(Q+\varepsilon Q')^3}\right)$$

$$\left.+\frac{(P+\varepsilon P')^2}{(Q+\varepsilon Q')}\left[\frac{2P'^2}{(Q+\varepsilon Q')^2} - \frac{8(P+\varepsilon P')P'Q'}{(Q+\varepsilon Q')^3} + \frac{6(P+\varepsilon P')^2Q'^2}{(Q+\varepsilon Q')^4}\right]\right\}$$

From here we set $\varepsilon = 0$,

$$= exp\frac{P^2}{Q^2}\left\{\frac{P^2}{Q}\left[\frac{2PP'}{Q^2} - \frac{2P^2Q'}{Q^3}\right]^2 + 2\left[\frac{2PP'}{Q^2} - \frac{2P^2Q'}{Q^3}\right]\left[\frac{2PP'}{Q} - \frac{2P^2Q'}{Q^2}\right]\right.$$

$$+\frac{2P'^2}{Q} - \frac{4PP'Q'}{Q^2} + \frac{2P^2Q'^2}{Q^3}$$

$$\left.+\frac{P^2}{Q}\left[\frac{2P'^2}{Q^2} - \frac{8PP'Q'}{Q^3} + \frac{6P^2Q'^2}{Q^4}\right]\right\}$$

$$= exp\frac{P^2}{Q^2}\left\{\left[\frac{2PP'}{Q^2} - \frac{2P^2Q'}{Q^3}\right]^2\left[\frac{P^2}{Q} + \frac{5Q}{2}\right] + \left[\frac{2PP'}{Q^2} - \frac{2P^2Q'}{Q^3}\right]\frac{P'Q - PQ'}{P}\right\}$$

$$= e^{\frac{P^2}{Q^2}}\left[\frac{2PP'}{Q^2} - \frac{2P^2Q'}{Q^3}\right]^2\left[\frac{Q^4 + 2P^4 + 5Q^2P^2}{2P^2Q}\right]. \qquad \textbf{(2. 3)}$$

Thus we have shown that second derivative of $\mathcal{L}_2$ is always positive and thus satisfy [C4].

The same method is applied on F_1 and we get

$$\frac{d^2F_1}{d\varepsilon^2} = \left\{\left[\frac{P'Q - PQ'}{Q^2}\right]\left[2\mathrm{R}' - \frac{2\mathrm{RQ}'}{Q}\right] + \mathrm{R}\left[\frac{P'Q - PQ'}{Q^2}\right]^2\right\} e^{\frac{\mathrm{P}}{\mathrm{Q}}} .$$

Setting $R = Q$ for F_1

$$\frac{d^2F_1}{d\varepsilon^2} = \frac{(P'Q - PQ')^2}{Q^3} e^{\frac{\mathrm{P}}{\mathrm{Q}}} , \tag{2. 4}$$

and for F_3

$$\frac{d^2F_3}{d\varepsilon^2} = e^{-\frac{Q}{P}}\left\{\left(\frac{Q'P - QP'}{P^2}\right)^2\left[\left(1 + \frac{P}{Q}\right)^2 + \frac{P^2}{Q^2}\right]\frac{P^2}{Q}\right\}. \tag{2. 5}$$

Hence we have shown that both F_1 and F_3 satisfy the minimization condition [C4]. The detailed workings of the minimization are given in appendix B.

2.4 Explicit Example

In this section we will be looking at the 3 specific generalizations of the linear Dirac equations which satisfy the constraints [C1] to [C4]. As mentioned earlier our Q will take the form $\bar{\psi}A_\mu\gamma^\mu\psi$ where $A_\mu = (A, 0,0,0)$, is a time-like constant background field resulting in our Lorentz violation. P is chosen to take the form $P = [i\bar{\psi}\gamma^\mu\partial_\mu\psi - i(\partial_\mu\bar{\psi})\gamma^\mu\psi]/2$. Hence our 3 models are

$$\mathcal{L}_1 = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \bar{\psi}A_\mu\gamma^\mu\psi \exp\frac{i\bar{\psi}\gamma^\mu\partial_\mu\psi - i(\partial_\mu\bar{\psi})\gamma^\mu\psi}{2\bar{\psi}A_\mu\gamma^\mu\psi} \tag{2. 6}$$

$$\mathcal{L}_2 = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

$$+\frac{\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi - i(\partial_\mu\bar{\psi})\gamma^\mu\psi\right)^2}{4\bar{\psi}A_\mu\gamma^\mu\psi}\exp\frac{\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi - i(\partial_\mu\bar{\psi})\gamma^\mu\psi\right)^2}{4\left(\bar{\psi}A_\mu\gamma^\mu\psi\right)^2} \qquad \textbf{(2. 7)}$$

$$\mathcal{L}_3 = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

$$+\frac{\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi - i(\partial_\mu\bar{\psi})\gamma^\mu\psi\right)^2}{4\bar{\psi}A_\mu\gamma^\mu\psi}\exp\left[-\frac{2\bar{\psi}A_\mu\gamma^\mu\psi}{i\bar{\psi}\gamma^\mu\partial_\mu\psi - i(\partial_\mu\bar{\psi})\gamma^\mu\psi}\right]. \qquad \textbf{(2. 8)}$$

Chapter 3

Plane-Wave Approximation and Modified Dispersion Relations

In this chapter, we will find plane wave solutions to the nonlinear Dirac equations. As with the linear equation, we want the solution to be eigenstates of both momentum and energy. In Schrödinger representation of momentum $\hat{\boldsymbol{p}} = -i\hbar\boldsymbol{\partial}$[3], we have the expression

$$\hat{\boldsymbol{p}}\psi_p = p\psi_p \,. \qquad \textbf{(3. 1)}$$

The energy-eigenvalue equation is given by

$$-i\hbar\partial_t\psi_E = E\psi_E \,. \qquad \textbf{(3. 2)}$$

We look for plane-wave solutions of the form

$$\psi(\boldsymbol{x}, t) = e^{ik\cdot x}u(k) \,. \qquad \textbf{(3. 3)}$$

where k_μ is the four vector and we set $\hbar = c = 1$.

3.1 Derivation of plane wave solutions

As the x dependence is confined to the exponent, we have

[3] Here $\hat{\boldsymbol{p}}$ represents the 3 momentum.

$$\partial_\mu \psi = -ik_\mu e^{ik\cdot x} u \,. \quad \textbf{(3. 4)}$$

Substituting it into the Dirac equation and simplifying we will get

$$(\gamma^\mu k_\mu - m)u = \begin{pmatrix} E-M & -k\cdot\vec{\sigma} \\ k\cdot\vec{\sigma} & -E-M \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} (E-M)u_A - (k\cdot\vec{\sigma})u_B \\ (k\cdot\vec{\sigma})u_A - (E+M)u_B \end{pmatrix} = 0\,, \quad \textbf{(3. 5)}$$

where u_A and $u_B$represent the upper two and lower two components respectively.

We then obtain the expression for u_A and u_B which are given by

$$u_A = \frac{k\cdot\vec{\sigma}}{E-M}\, u_B\,, \qquad u_B = \frac{k\cdot\vec{\sigma}}{E+M}\, u_A$$

and from substituting one to the other

$$u_A = \frac{(k\cdot\vec{\sigma})^2}{E^2-M^2}\, u_A\,. \quad \textbf{(3. 6)}$$

Evaluating $(k\cdot\vec{\sigma})^2$

$$k\cdot\vec{\sigma} = k_x\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + k_y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + k_z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} k_z & k_x - ik_y \\ k_x + ik_y & -k_z \end{pmatrix}$$

$$(k\cdot\vec{\sigma})^2 = \begin{pmatrix} k_z^2 + (k_x - ik_y)(k_x + ik_y) & k_z(k_x - ik_y) - k_z(k_x - ik_y) \\ k_z(k_x + ik_y) - k_z(k_x + ik_y) & k_z^2 + (k_x - ik_y)(k_x + ik_y) \end{pmatrix}$$

$$= k^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad \textbf{(3. 7)}$$

Thus

$$u_A = \frac{k^2}{E^2-M^2}\, u_A\,. \quad \textbf{(3. 8)}$$

Here we will get back our energy dispersion relation

$$E^2 - M^2 = k^2\,. \quad \textbf{(3. 9)}$$

To get the plane wave solutions we consider four cases, letting $u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We then substitute in the corresponding energy E and normalize the spinors. Finally we will end up with the plane wave solutions

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{k_z}{E+M} \\ \frac{k_x + ik_y}{E+M} \end{pmatrix}, \qquad u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{k_x - ik_y}{E+M} \\ -\frac{k_z}{E+M} \end{pmatrix},$$

$$u^{(3)} = N \begin{pmatrix} \frac{k_z}{E-M} \\ \frac{k_x + ik_y}{E-M} \\ 1 \\ 0 \end{pmatrix}, \qquad u^{(4)} = N \begin{pmatrix} \frac{k_x - ik_y}{E-M} \\ \frac{k_z}{E-M} \\ 0 \\ 1 \end{pmatrix}, \tag{3. 10}$$

with $N = \sqrt{|E| + M}$

3.2 Explicit Modified Energy Dispersion Relation

From our modified Dirac Lagrangian, we will apply the plane wave solution to get the modified energy dispersion relation. Here we will start with $\mathcal{L}_1$ and apply the Euler-Lagrangian equation $\left(\left(\frac{\partial \mathcal{L}}{\partial \bar{\psi}}\right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})}\right) = 0\right)$ to obtain our modified energy dispersion relation.

$$\mathcal{L}_1 = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + \bar{\psi} A_\mu \gamma^\mu \psi \exp \frac{i\bar{\psi}\gamma^\mu \partial_\mu \psi - i(\partial_\mu \bar{\psi})\gamma^\mu \psi}{2\bar{\psi} A_\mu \gamma^\mu \psi} \tag{3. 11}$$

$$\frac{\partial \mathcal{L}_1}{\partial \bar{\psi}} = \left\{ A_\mu \gamma^\mu \psi + \bar{\psi} A_\mu \gamma^\mu \psi \left[\frac{i\gamma^\mu \partial_\mu \psi}{2\bar{\psi} A_\mu \gamma^\mu \psi} - \frac{\left(i\bar{\psi}\gamma^\mu \partial_\mu \psi - i\left(\partial_\mu \bar{\psi}\right)\gamma^\mu \psi\right) A_\mu \gamma^\mu \psi}{2\left(\bar{\psi} A_\mu \gamma^\mu \psi\right)^2} \right] \right\}$$

$$\cdot \exp\left(\frac{i\bar{\psi}\gamma^\mu \partial_\mu \psi - i\left(\partial_\mu \bar{\psi}\right)\gamma^\mu \psi}{2\bar{\psi} A_\mu \gamma^\mu \psi} \right) + \left(i\gamma^\mu \partial_\mu - m\right)\psi \qquad \textbf{(3.12)}$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}_1}{\partial(\partial_\mu \bar{\psi})} \right) = -\frac{i\gamma^\mu \partial_\mu \psi}{2} \cdot \exp\left(\frac{i\bar{\psi}\gamma^\mu \partial_\mu \psi - i\left(\partial_\mu \bar{\psi}\right)\gamma^\mu \psi}{2\bar{\psi} A_\mu \gamma^\mu \psi} \right) \qquad \textbf{(3. 13)}$$

(Using $\mathrm{i}\gamma^\mu \partial_\mu \psi = m\psi$, $-\mathrm{i}\gamma^\mu \partial_\mu \bar{\psi} = m\bar{\psi}$, $\frac{\psi^\dagger \psi}{\bar{\psi}\psi} = \frac{E}{m}$)[4]

$$\left(\frac{\partial \mathcal{L}_1}{\partial \bar{\psi}} \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \bar{\psi}\right)} \right) = 0 \qquad \textbf{(3. 14)}$$

$$\left[\mathrm{A}_\mu \gamma^\mu \psi - \frac{m^2}{E}\gamma^\mu \psi + m\psi \right] e^{\frac{m^2}{AE}} + \left(\gamma^\mu k_\mu - \mathrm{m}\right)\psi = 0$$

$$\left[\gamma^\mu \left(k_\mu + A_\mu e^{\frac{m^2}{AE}} - \frac{m^2}{E} e^{\frac{m^2}{AE}} \right) - \left(m - m e^{\frac{m^2}{AE}} \right) \right] \psi = 0 \,.$$

Squaring both terms we will obtain the expression

$$k^2 + 2AEe^{\frac{m^2}{AE}} - 2m^2 e^{\frac{m^2}{AE}} + A^2 e^{\frac{2m^2}{AE}} - \frac{2Am^2}{E} e^{\frac{2m^2}{AE}} + \frac{m^4}{E^2} e^{\frac{2m^2}{AE}}$$

$$= m^2 - 2m^2 e^{\frac{m^2}{AE}} + m^2 e^{\frac{2m^2}{AE}} \,. \qquad \textbf{(3. 15)}$$

Here we require the information measure $\bar{\psi} A_\mu \gamma^\mu \psi \exp \frac{i\bar{\psi}\gamma^\mu \partial_\mu \psi - i(\partial_\mu \bar{\psi})\gamma^\mu \psi}{2\bar{\psi} A_\mu \gamma^\mu \psi}$ to be small, that is A is small. We also assume that the order of A is the same as m. Hence under this assumption we can safely ignore all higher order of A. Also we make the Taylor approximation for $e^{\frac{2m^2}{AE}} \approx 1 + \frac{m^2}{AE}$. Hence,

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[4] $\mathrm{i}\gamma^\mu \partial_\mu \psi = m\psi$ and $-\mathrm{i}\gamma^\mu \partial_\mu \bar{\psi} = m\bar{\psi}$ follows from linear Dirac equation. $\frac{\psi^\dagger \psi}{\bar{\psi}\psi} = \frac{E}{m}$ can be derived from plane wave solution and is shown in appendix.

$$k^2 = m^2 + \left(1 + \frac{m^2}{AE}\right)\left[-2AE + m^2\left(1 + \frac{m^2}{AE}\right)\right]$$

$$= m^2 - 2AE - 2m^2 + m^2\left(1 + \frac{2m^2}{AE} + \frac{m^4}{A^2E^2}\right)$$

$$= -2AE + \frac{2m^4}{AE} + \frac{m^6}{A^2E^2}. \qquad \textbf{(3. 16)}$$

The resulting expression $k^2 = -2AE + \frac{2m^4}{AE} + \frac{m^6}{A^2E^2}$ is our modified energy dispersion relation.

The same method is applied to $\mathcal{L}_2$ and $\mathcal{L}_3$. Below is a summary of their results.

- $\mathcal{L}_1$: $k^2 = -2AE + \frac{2m^4}{AE} + \frac{m^6}{A^2E^2}$ **(3. 17)**
- $\mathcal{L}_2$: $k^2 = m^2 + \frac{2m^4}{AE}$ **(3. 18)**
- $\mathcal{L}_3$: $k^2 = m^2 + m^2 e^{\frac{2AE}{m^2}}$ **(3. 19)**

The detailed derivation of the modified dispersion relation is shown in Appendix C.

Chapter 4

Neutrino Oscillations

Neutrinos are of interest as they are weakly interacting and hence are valuable probe of new physics. They are observed to change flavor and this phenomena is termed neutrino oscillation because the probability of measuring a flavor varies periodically when they propagates. In the conventional theory, neutrinos are assigned constant mass to explain for neutrino oscillation. Other theories such as a Lorentz violating dispersion relation do not seem to be possible explanations of the leading order effect [2].

In this chapter, we will apply our modified energy dispersion relation derived from the generalized Dirac equation to neutrino oscillation and determine how well it agrees with conventional theory and also if there are new underlying physics. We will investigate the effect of our modified Dirac equation in the regime of high energy as it was mentioned is Ref. [14] that quantum nonlinearities and Lorentz violations might be related. We also do not reject the possibility that the nonlinearity is an effective nonlinearity summarizing the unknown microscopic physics rather than a fundamental modification to the quantum theory.

Before we go on to apply the modified energy dispersion relation, let us briefly go through the conventional theory to remind ourselves of the role of energy dispersion relation in calculating the probability of neutrino oscillations.

4.1 Conventional Theory

In the conventional theory, neutrinos having mass means that there exists neutrino mass eigenstates $v_i, i = 1,2, \ldots,$ each with a mass m_i. Considering the reaction $W^+ \rightarrow v_i + \bar{l}_\alpha$, where $\alpha = e,\ \mu,\ \tau$, what lepton mixing means is that the accompanying neutrino mass eigenstates is not always the same v_i. We can denote the amplitude for the W^+decay to produce specific combination of $v_i + \bar{l}_\alpha$ by $U^*_{\alpha i}$. Hence we will end up with the expression

$$\psi_\alpha(x) = \sum_i U^*_{\alpha i} \psi_i(x)\,, \tag{4. 1}$$

where $\psi_\alpha(x)$ are the neutrinos flavor eigenfunctions and $\psi_i(x)$ are the mass eigenfunctions. For two neutrino flavor oscillation, the mixing matrix is given by

$$U = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}. \tag{4. 2}$$

Now we consider a neutrino flavor change as depicted in the figure below.

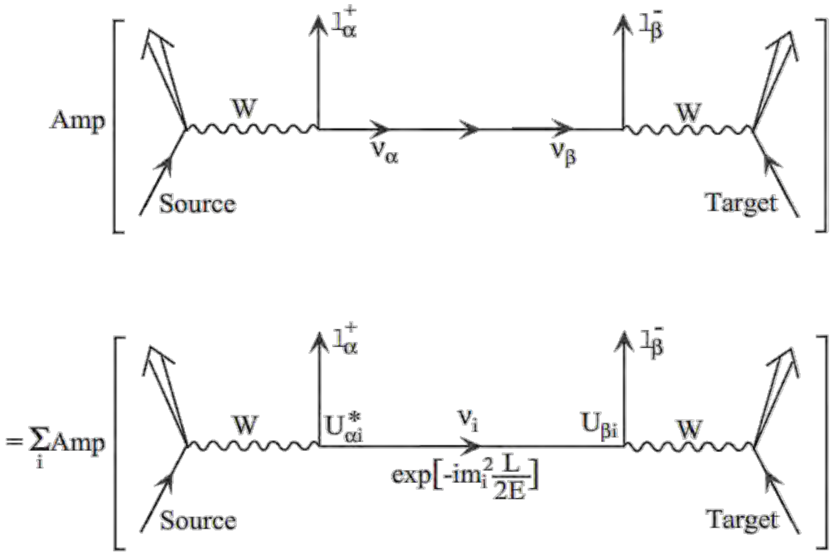

Figure 4.1 Neutrino flavour change (oscillation) in vacuum. 'Amp' denotes amplitude [12]

A neutrino is created at the source with a lepton $\bar{l}_\alpha$ and it travels a certain distance L before interacting at the detector to produce another lepton l_β. Hence as it propagates from the source to the detector, it changes from v_α to v_β.

The amplitude of such a reaction is given by

$$Amp(v_\alpha \to v_\beta) = \sum_i U^*_{\alpha i} Prop(v_i) U_{\beta i} . \quad \textbf{(4. 3)}$$

To find $Prop(v_i)$, consider rest frame of v_i, where time in that frame is τ_i and v_i have mass m_i. Here by solving the Schrödinger equation we will obtain the solution

$$\psi_i(\tau_i) = e^{-im_i\tau_i}\psi_i(0) , \quad \textbf{(4. 4)}$$

$$Prop(v_i) = e^{-im_i\tau_i} . \quad \textbf{(4. 5)}$$

Now introducing lab frame variables L, t, E_i and p_i for the distance the neutrino travelled, the time elapsed in lab frame, energy and momentum of the neutrinos respectively. Here there is no Lorentz violation and hence

$$m_i\tau_i = E_i t - p_i L . \quad \textbf{(4. 6)}$$

Here we will use the energy dispersion relation to obtain

$$p_i = \sqrt{E^2 - m_i^2} \cong E - \frac{m_i^2}{2E} . \quad \textbf{(4. 7)}$$

Giving us the probability of flavor change

$$P_{v_\alpha \to v_\beta}(E, L) = \sin^2 2\theta \sin^2\left(\frac{L}{2}\Delta p\right) , \quad \textbf{(4. 8)}$$

where $\Delta p = \Delta m^2/2E$ and $\Delta m^2 = m_i^2 - m_j^2$. Maximum oscillation occurs when

$$\frac{L_0}{2}\Delta p = \frac{\pi}{2} . \quad \textbf{(4. 9)}$$

After restoring the $\hbar's$ and $c's$ we will finally get the oscillation length L_0 given by

$$L_0 = \frac{2\pi\hbar c E}{\Delta m^2 c^4}\,. \qquad \textbf{(4. 10)}$$

The oscillation length expressions are valid even for our modified dispersion relation but with Δp taking a different form.

4.2 Generalized Dirac Equation in Neutrino Oscillation

In this section, we will be applying our three generalized Dirac equation into neutrino oscillation. We will first start off by briefly discussing the Lorentz violating parameter along with the experimental data that will be used. Then we will apply our modified energy dispersion relation to neutrino oscillation. The results will be compared against experimental data to check that they are consistent. Lastly we will interpret the results and compare their differences with the conventional theory.

Here we wish to introduce a dimensionless nonlinear/Lorentz violating parameter that makes our equation nonlinear. From Ref. [11], we can take the size of that parameter f to be

$$f \sim 10^{-27}\,. \qquad \textbf{(4. 11)}$$

Here we also assume that this parameter may be dependent on neutrino species and it's order is approximated to be $\Delta f \sim f$ .This parameter will be compared with the magnitude of the nonlinear terms in our results and they should be consistent.

For the experimental data used, the relevant data are

$$\Delta m^2 = 2.5 \times 10^{-3}\ eV^2 \qquad \textbf{(4. 12)}$$

$$E = 100\ GeV\,, \qquad \textbf{(4. 13)}$$

that is the mean energy of the neutrino beam will be 100 GeV. Here the Δm^2 is from the atmospheric neutrino sector. Also we make the assumption that the order of magnitude of the mass of neutrino approximately equal to the root of the difference between the masses

$$m \approx \sqrt{\Delta m^2}\,. \tag{4. 14}$$

4.2.1 Lagrangian 1

Starting with $\mathcal{L}_1$, the result we obtained earlier was $k^2 = -2AE + \frac{2m^4}{AE} + \frac{m^6}{A^2E^2}$. Comparing it to the conventional energy dispersion expression, $k^2 = m^2$, we can see that m^2 term is removed and replaced with function of our background field A, energy of neutrino $E, m^4$and m^6. Here we make a hypothesis that neutrinos do not have mass, instead the different species of neutrinos interact differently with the background field. This interaction is then responsible for the phenomena of neutrino oscillation. Hence with this hypothesis, we will now proceed to make momentum p the subject so that it can be substituted into our oscillation length expression.

$$k^2 = -2AE + \frac{2m^4}{AE} + \frac{m^6}{A^2E^2} \approx -2AE$$

$$E^2 - p^2 = -2AE$$

$$p = \sqrt{E^2 - 2AE} \approx E - A \tag{4. 15}$$

$$\Delta p = \Delta A\,. \tag{4. 16}$$

Thus our new oscillation length L_1 will be

$$L_1 = \frac{\pi\hbar}{c^3}\Delta p = \frac{\pi\hbar}{c^3}\Delta A\,. \tag{4. 17}$$

Now setting our oscillation length to be consistent with experimental results,

$$L_0 = L_1 \rightarrow \Delta m^2 = 2E\Delta A \quad \mathbf{(4.18)}$$

where Δm^2 is the experimentally obtained mass difference. By substituting in the experimental values states in (4. 12) and (4. 13) we find that

$$\Delta A \approx \frac{2.5 \times 10^{-3}}{2 \times 10^{11}} = 1.25 \times 10^{-14}\, eV\,. \quad \mathbf{(4.19)}$$

Consider the order of magnitude comparison with (4. 11) , we have

$$\Delta A \approx \Delta(f_1 E) \quad \mathbf{(4.20)}$$

$$f_1 \approx 10^{-25}\,, \quad \mathbf{(4.21)}$$

which is only 2 order of magnitude different from (4. 11).

Hence instead of different neutrinos species having different mass eigenstates, lagrangian 1 use the idea of a background field interacting differently with the different species to explain neutrino oscillation. The order of magnitude of the differences between the interactions of the field with different species of neutrinos agrees quite well with experimental data.

4.2.2 Lagrangian 2

From $k^2 = m^2 + \frac{2m^4}{AE}$, we see that the modified energy dispersion relation is similar to that of the conventional theory except with the addition of a term $\frac{2m^4}{AE}$. From $\mathcal{L}_2$ (3.18) we note that $1/A$ must be small in order for the information measure to be small. Hence this modified energy dispersion relation can be seen as adding a small nonlinear/Lorentz violating term to the conventional energy dispersion expression. Hence from (4. 10)

$$L_0' = \frac{2\pi\hbar c E}{\Delta m^2(1+X)c^4}, \quad \textbf{(4. 22)}$$

where X is a small change and to be determined. Now making momentum the subject and substituting the expression into our (4. 10)

$$p = \sqrt{E^2 - m^2 - \frac{2m^4}{AE}} \approx E - \frac{m^2}{2E} - \frac{m^4}{AE^2}. \quad \textbf{(4. 23)}$$

From here we can have two different ways of interpreting our results. In the first case we could let different neutrino species have the same mass but interacts differently with our background field A. Hence we will end up with the expression

$$\Delta p = \frac{m^4}{E^2 \Delta A}. \quad \textbf{(4. 24)}$$

Here we make the assumption that m^4 have the same order of magnitude as $(\Delta m^2)^2$. We then substitute the known values of E^2 and m^4 and let $X \approx 10^{-27}$ as required by (4. 11). Thus

$$X_2 = \frac{2E\Delta p}{\Delta m^2} - 1 = -\frac{2(2.5\times10^{-3})^2}{10^{11}\times 2.5\times 10^{-3}}\frac{1}{\Delta A} \approx 10^{-27}$$

$$\Delta A \approx 5\times 10^{-14}\ eV, \quad \textbf{(4. 25)}$$

which is in good agreement with what we have found earlier.

For the second case, we let the neutrino mass be different for each of the different species and the interaction with the background field be the same. Thus we will obtain the expression

$$\Delta p = -\frac{\Delta m^2}{2E} - \frac{\Delta m^4}{AE^2}. \quad \textbf{(4. 26)}$$

Here we make the assumption that $\Delta m^4 \approx (\Delta m^2)^2$. We then substitute the known values of Δm^2 and Δm^4 let $X \approx 10^{-27}$ as required by (4.11).

$$X_2 = \frac{2E\Delta p}{\Delta m^2} - 1 = -\frac{2(2.5 \times 10^{-3}) - \frac{2(2.5 \times 10^{-3})^2}{AE}}{\Delta m^2} \approx 10^{-27}$$

$$A \approx 2.5 \times 10^{-14}\ eV\ . \qquad \textbf{(4. 27)}$$

Hence for both interpretation of $\mathcal{L}_2$, the magnitude of the background field A was found to be 10^{-14} which agrees with $\mathcal{L}_1$.

The results can be interpreted as such. In the first case, besides having mass, the neutrinos interact differently with the background field. Thus the background field also contributes to the neutrino oscillation. In the second case, the different species of neutrinos have different masses just as in the conventional theory. However, apart from the difference in masses, there is also a small contribution due to interactions with the background field.

These model derived from $\mathcal{L}_2$ have the advantage in that it is possible to find the individual mass of the neutrinos due to the existence of m^2 and m^4terms, unlike the conventional theory whereby only the difference between the mass of the neutrinos are known.

4.2.3 Lagrangian 3

For $\mathcal{L}_3$, $k^2 = m^2 + m^2 e^{2AE/m^2}$. $\mathcal{L}_3$ is similar to $\mathcal{L}_2$ in that both of them have similar form with the conventional energy dispersion relation adding a small nonlinear/ Lorentz violating term. Following the steps above, we have the momentum expression as

$$p = \sqrt{E^2 - m^2(1 + e^{\frac{2AE}{m^2}})} \approx E - \frac{m^2}{2E}(1 + e^{\frac{2AE}{m^2}})$$

$$\Delta p = \frac{1}{2E}\left[\Delta m^2 + m_1^2 e^{\frac{2A_1E}{m_1^2}} - m_2^2 e^{\frac{2A_2E}{m_2^2}}\right], \quad \textbf{(4. 28)}$$

and the corresponding X_3 value is given by

$$X_3 = \frac{m_1^2 e^{\frac{2A_1E}{m_1^2}} - m_2^2 e^{\frac{2A_2E}{m_2^2}}}{\Delta m^2}.$$

As this expression cannot be solved analytically, we will have to make some order of magnitude approximation. Like in $\mathcal{L}_2$, we have 2 ways of interpreting the results.

In the first case, we assume that the different species of neutrinos all have the same mass but interact different with the background field A, i.e $m_1 = m_2$, $A_1 \neq A_2$. Thus we will end up with the expression

$$X_3 = e^{\frac{2A_1E}{m^2}} - e^{\frac{2A_2E}{m^2}} \approx 10^{-25}\,. \quad \textbf{(4. 29)}$$

Recalling that $1/A$ is small and hence A is large for $\mathcal{L}_3$, we will end up with the condition that ΔA must be very small such that $e^{\frac{2A_1E}{m^2}} - e^{\frac{2A_2E}{m^2}} \approx 10^{-25}$.

Thus in summary, this case postulates that all neutrinos have the same mass and they interact strongly with the background field A. And also the interactions with the background field differ slightly for each neutrino species.

For case two, we assume that both the mass and interaction with the background field varies with the neutrino species. Meaning $m_1 \neq m_2$, and $A_1 \neq A_2$. Thus we have

$$X_3 = \frac{m_1^2 e^{\frac{2A_1E}{m_1^2}} - m_2^2 e^{\frac{2A_2E}{m_2^2}}}{\Delta m^2} \approx 10^{-25} \quad \textbf{(4. 30)}$$

$$m_1^2 e^{\frac{2A_1E}{m_1^2}} - m_2^2 e^{\frac{2A_2E}{m_2^2}} \approx 10^{-25} \times 2.5 \times 10^{-3} = 2.5 \times 10^{-28}\,. \qquad \textbf{(4. 31)}$$

And since we know that $m_1^2 - m_2^2 = \Delta m^2 = 2.5 \times 10^{-3}$, we can see from the above expression that $\Delta A/m$ must be very small.

The interpretation of this case will be that different species of neutrino have different mass and they also interact differently with the background field. However the difference in the combination of the interaction divided by the mass is very small between the different species.

Chapter 5

Future Aims

In this chapter, we will discuss some future possible expansion of this theory along with some of the possible application of the models we have derived earlier. Firstly in our discussion above, we set four constraints and created the models based on them. The four constraints are namely homogeneity, uncertainty, positivity and minimization. For future development of this theory, we can try relaxing one of the constraints while adding another to obtain other models. For example, we can relax the constraint on homogeneity and instead require our generalized Dirac equation to be Lorentz invariant. Hence the final resulting model will be positive definite, contains derivatives, is minimum and Lorentz invariant but not invariant under the scaling $\psi \to \lambda\psi$.

In our discussion above, we have applied our generalized Dirac equation to two neutrinos case and attempted to find the scale of our Lorentz violating background field. A possible future extension of this project will be to apply the three models we obtained earlier to the case of three neutrinos. From there we can obtain more information of our models and compare it to the case of two neutrinos we used earlier and see if they agree with each other.

Using our nonlinear Dirac equation, we have obtained a modified energy dispersion relation. This nonlinearity may have interesting application to study of dark matter and sterile neutrinos. Also at high energy nonlinearity becomes of greater significance, we can attempt to apply it to baryogensis where it could be a new source of C and CP violation. However more research is to be done before we can apply it to these areas.

Chapter 6

Conclusion

In this thesis, we have obtained a nonlinear generalization of the Dirac equation. The approach we used was the application of information theoretic axiom to constrain our generalization. After applying the homogeneity, uncertainty, positivity and minimization constraints, the resulting generalization is found to be nonlinear and Lorentz violating. Using the constraints, we are able to generate various possible generalizations. Here we looked at three generalizations which satisfy the various constraints and they are non-local.

Using the generalizations, we proceeded to derive modified energy dispersion relations using the plane wave solutions. The modified energy dispersions are then applied to neutrino oscillation to examine how the probabilities of neutrino oscillation changes. From there we can predict possible future modifications to neutrino oscillation experiments. Some of the advantages of our generalized Dirac equations include the possibility of massless neutrinos or the possibility of finding individual mass of neutrinos instead of the difference between them.

We have also discussed the possibility of changing our information theoretic constraints to generate new generalizations. Applications of our generalized Dirac equation to various field of physics is possible. If the generalizations serve as fundamental equations, we can use it as a test for quantum linearity in various areas of physics. If instead it is an effective equation, we can see it as a way to describe various physics phenomena.

Appendix A Notations

Here we list some of the conventions used for the convenience of the reader.

We work in 3+1 dimensional flat space with metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{A. 1}$$

The Dirac Equation is

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\phi = 0 \tag{A. 2}$$

Where the gamma matrices satisfy

$$\{\gamma^{\mu}, \gamma^{\nu}\} = \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu} \tag{A. 3}$$

The Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A. 4}$$

The Dirac representation of the γ matrices are

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{A. 5}$$

Appendix B Minimization

Lagrangian 1

$$L_1 = Re^{P/Q} \quad \textbf{(B. 1)}$$

$$\frac{dL_1}{d\varepsilon} = R'e^{\frac{P+\varepsilon P'}{Q+\varepsilon Q'}} + (R+\varepsilon R')\left[\frac{P'}{Q+\varepsilon Q'} - \frac{(P+\varepsilon P')Q'}{(Q+\varepsilon Q')^2}\right]e^{\frac{P+\varepsilon P'}{Q+\varepsilon Q'}} \quad \textbf{(B. 2)}$$

$$\frac{d^2L_1}{d\varepsilon^2} = R'\left[\frac{P'}{Q+\varepsilon Q'} - \frac{(P+\varepsilon P')Q'}{(Q+\varepsilon Q')^2}\right]e^{\frac{P+\varepsilon P'}{Q+\varepsilon Q'}} + R'\left[\frac{P'}{Q+\varepsilon Q'} - \frac{(P+\varepsilon P')Q'}{(Q+\varepsilon Q')^2}\right]e^{\frac{P+\varepsilon P'}{Q+\varepsilon Q'}}$$

$$+ (R+\varepsilon R')\left[\frac{P'}{Q+\varepsilon Q'} - \frac{(P+\varepsilon P')Q'}{(Q+\varepsilon Q')^2}\right]e^{\frac{P+\varepsilon P'}{Q+\varepsilon Q'}}$$

$$+ (R+\varepsilon R')e^{\frac{P+\varepsilon P'}{Q+\varepsilon Q'}}\left[-\frac{2P'Q'}{(Q+\varepsilon Q')^2} + \frac{2(P+\varepsilon P')Q'^2}{(Q+\varepsilon Q')^3}\right]$$

$$= e^{\frac{P}{Q}}\left\{2R'\left[\frac{P'}{Q} - \frac{PQ'}{Q^2}\right] + R\left[\frac{P'}{Q} - \frac{PQ'}{Q^2}\right]^2 + R\left[\frac{2PQ'^2}{Q^3} - \frac{2P'Q'}{Q^2}\right]\right\}$$

$$= e^{\frac{P}{Q}}\left\{\left[\frac{P'}{Q} - \frac{PQ'}{Q^2}\right]\left[2R' - \frac{2RQ'}{Q}\right] + R\left[\frac{P'}{Q} - \frac{PQ'}{Q^2}\right]^2\right\}$$

(Here we set R=Q)

$$= e^{\frac{P}{Q}}\left\{\left[\frac{P'}{Q} - \frac{PQ'}{Q^2}\right]\left[2Q' - \frac{2QQ'}{Q}\right] + Q\left[\frac{P'}{Q} - \frac{PQ'}{Q^2}\right]^2\right\}$$

$$= \frac{(P'Q - PQ')^2}{Q^3}e^{\frac{P}{Q}} \quad \textbf{(B. 3)}$$

Thus this Lagrangian is always positive when we extremise it if we set $R(\psi,\bar{\psi}) = Q(\psi,\bar{\psi})$, fulfilling the constraint [C4].

Lagrangian 2

$$L_2 = \frac{P^2}{Q} e^{\frac{P^2}{Q^2}} \quad \textbf{(B. 4)}$$

$$\frac{dL_2}{d\varepsilon} = e^{\frac{P^2}{Q^2}} \left[\frac{2(P+\varepsilon P')P'}{(Q+\varepsilon Q')} - \frac{(P+\varepsilon P')^2 Q'}{(Q+\varepsilon Q')^2} + \frac{(P+\varepsilon P')^2}{(Q+\varepsilon Q')} \left(\frac{2(P+\varepsilon P')P'}{(Q+\varepsilon Q')^2} - \frac{2(P+\varepsilon P')^2 Q'}{(Q+\varepsilon Q')^3} \right) \right] \quad \textbf{(B. 5)}$$

$$\frac{d^2L_2}{d\varepsilon^2} = e^{\frac{P(\psi,\bar{\psi})^2}{Q(\psi,\bar{\psi})^2}} \left\{ \left(\frac{2(P+\varepsilon P')P'}{(Q+\varepsilon Q')^2} - \frac{2(P+\varepsilon P')^2 Q'}{(Q+\varepsilon Q')^3} \right) \left[\frac{2(P+\varepsilon P')P'}{(Q+\varepsilon Q')} - \frac{(P+\varepsilon P')^2 Q'}{(Q+\varepsilon Q')^2} + \frac{(P+\varepsilon P')^2}{(Q+\varepsilon Q')} \left(\frac{2(P+\varepsilon P')P'}{(Q+\varepsilon Q')^2} - \frac{2(P+\varepsilon P')^2 Q'}{(Q+\varepsilon Q')^3} \right) \right] + \frac{2P'^2}{Q+\varepsilon Q'} - \frac{4(P+\varepsilon P')P'Q'}{(Q+\varepsilon Q')^2} + \frac{2(P+\varepsilon P')^2 Q'^2}{(Q+\varepsilon Q')^3} + \left(\frac{2(P+\varepsilon P')P'}{Q+\varepsilon Q'} - \frac{2(P+\varepsilon P')^2 Q'}{(Q+\varepsilon Q')^2} \right) \left(\frac{2(P+\varepsilon P')P'}{(Q+\varepsilon Q')^2} - \frac{2(P+\varepsilon P')^2 Q'}{(Q+\varepsilon Q')^3} \right) + \frac{(P+\varepsilon P')^2}{(Q+\varepsilon Q')} \left[\frac{2P'^2}{(Q+\varepsilon Q')^2} - \frac{8(P+\varepsilon P')P'Q'}{(Q+\varepsilon Q')^3} + \frac{6(P+\varepsilon P')^2 Q'^2}{(Q+\varepsilon Q')^4} \right] \right\}$$

$$= e^{\frac{P^2}{Q^2}} \left\{ \frac{P^2}{Q} \left[\frac{2PP'}{Q^2} - \frac{2P^2Q'}{Q^3} \right]^2 + 2 \left[\frac{2PP'}{Q^2} - \frac{2P^2Q'}{Q^3} \right] \left[\frac{2PP'}{Q} - \frac{2P^2Q'}{Q^2} \right] + \frac{2P'^2}{Q} - \frac{4PP'Q'}{Q^2} + \frac{2P^2Q'^2}{Q^3} + \frac{P^2}{Q} \left[\frac{2P'^2}{Q^2} - \frac{8PP'Q'}{Q^3} + \frac{6P^2Q'^2}{Q^4} \right] \right\}$$

$$= e^{\frac{P^2}{Q^2}} \left\{ \left[\frac{2PP'}{Q^2} - \frac{2P^2Q'}{Q^3} \right]^2 \left[\frac{P^2}{Q} + \frac{5Q}{2} \right] + \left[\frac{2PP'}{Q^2} - \frac{2P^2Q'}{Q^3} \right] \frac{P'Q - PQ'}{P} \right\}$$

$$= e^{\frac{P^2}{Q^2}}\left[\frac{2PP'}{Q^2} - \frac{2P^2Q'}{Q^3}\right]^2\left[\frac{Q^4 + 2P^4 + 5Q^2P^2}{2P^2Q}\right] \quad \textbf{(B. 6)}$$

Thus it is always positive.

Lagrangian 3

$$L_3 = \frac{P^2}{Q}e^{-\frac{Q}{P}} \quad \textbf{(B. 7)}$$

$$\frac{dL_3}{d\varepsilon} = e^{-\frac{Q+\varepsilon Q'}{P+\varepsilon P'}}[\frac{2(P+\varepsilon P')P'}{Q+\varepsilon Q'} - \frac{(P+\varepsilon P')^2Q'}{(Q+\varepsilon Q')^2} - \frac{(P+\varepsilon P')^2}{Q+\varepsilon Q'} \cdot \left(\frac{Q'}{P+\varepsilon P'} - \frac{(Q+\varepsilon Q')P'}{(P+\varepsilon P')^2}\right)] \quad \textbf{(B. 8)}$$

$$\begin{aligned}\frac{d^2L_3}{d\varepsilon^2} = e^{-\frac{Q+\varepsilon Q'}{P+\varepsilon P'}}\Bigg\{&-\left(\frac{Q'}{P+\varepsilon P'} - \frac{(Q+\varepsilon Q')P'}{(P+\varepsilon P')^2}\right)\Bigg[\frac{2(P+\varepsilon P')P'}{Q+\varepsilon Q'} - \frac{(P+\varepsilon P')^2Q'}{(Q+\varepsilon Q')^2} \\ &- \frac{(P+\varepsilon P')^2}{Q+\varepsilon Q'} \cdot \left(\frac{Q'}{P+\varepsilon P'} - \frac{(Q+\varepsilon Q')P'}{(P+\varepsilon P')^2}\right)\Bigg] + \frac{2P'^2}{Q+\varepsilon Q'} \\ &- \frac{2(P+\varepsilon P')P'Q'}{(Q+\varepsilon Q')^2} - \frac{2(P+\varepsilon P')P'Q'}{(Q+\varepsilon Q')^2} + \frac{2(P+\varepsilon P')^2Q'^2}{(Q+\varepsilon Q')^3} \\ &- \left(\frac{2(P+\varepsilon P')P'}{Q+\varepsilon Q'} - \frac{(P+\varepsilon P')^2Q'}{(Q+\varepsilon Q')^2}\right) \cdot \left(\frac{Q'}{P+\varepsilon P'} - \frac{(Q+\varepsilon Q')P'}{(P+\varepsilon P')^2}\right) \\ &- \frac{(P+\varepsilon P')^2}{Q+\varepsilon Q'} \cdot \left(-\frac{Q'P'}{(P+\varepsilon P')^2} - \frac{Q'P'}{(P+\varepsilon P')^2} + \frac{2(Q+\varepsilon Q')P'^2}{(P+\varepsilon P')^3}\right)\Bigg\}\end{aligned}$$

$$\begin{aligned}&= e^{-\frac{Q}{P}}\{\frac{P^2}{Q}\left(\frac{Q'}{P} - \frac{QP'}{P^2}\right)^2 - \left(\frac{Q'}{P} - \frac{QP'}{P^2}\right)\left[\frac{2PP'}{Q} - \frac{P^2Q'}{Q^2} + \frac{2PP'}{Q} - \frac{P^2Q'}{Q^2}\right] \\ &\quad + \frac{P^2}{Q}(\frac{2P'^2}{P^2} - \frac{4P'Q'}{PQ} + \frac{2Q'^2}{Q^2} + \frac{2P'}{P}(\frac{Q'}{P} - \frac{QP'}{P^2})\end{aligned}$$

$$= e^{-\frac{Q}{P}}\left\{\left(\frac{Q'}{P} - \frac{QP'}{P^2}\right)^2\left(\frac{P^2}{Q} + \frac{2P^3}{Q^2} + \frac{2P^4}{Q^3}\right)\right\}$$

$$= e^{-\frac{Q}{P}}\left\{\left(\frac{Q'}{P} - \frac{QP'}{P^2}\right)^2\left(\left(1 + \frac{P}{Q}\right)^2 + \frac{P^2}{Q^2}\right)\frac{P^2}{Q}\right\} \quad \textbf{(B. 9)}$$

Thus it is always positive.

Appendix C Energy dispersion relation

Lagrangian 1

$$\mathcal{L}_1 = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi + \bar{\psi}A_\mu\gamma^\mu\psi\exp\frac{i\bar{\psi}\gamma^\mu\partial_\mu\psi - i\left(\partial_\mu\bar{\psi}\right)\gamma^\mu\psi}{2\bar{\psi}A_\mu\gamma^\mu\psi} \qquad \textbf{(C. 1)}$$

$$\frac{\partial\mathcal{L}_1}{\partial\bar{\psi}} = \left\{A_\mu\gamma^\mu\psi + \bar{\psi}A_\mu\gamma^\mu\psi\left[\frac{i\gamma^\mu\partial_\mu\psi}{2\bar{\psi}A_\mu\gamma^\mu\psi} - \frac{\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi - i\left(\partial_\mu\bar{\psi}\right)\gamma^\mu\psi\right)A_\mu\gamma^\mu\psi}{2\left(\bar{\psi}A_\mu\gamma^\mu\psi\right)^2}\right]\right\}$$
$$\cdot \exp\left(\frac{i\bar{\psi}\gamma^\mu\partial_\mu\psi - i\left(\partial_\mu\bar{\psi}\right)\gamma^\mu\psi}{2\bar{\psi}A_\mu\gamma^\mu\psi}\right) + \left(i\gamma^\mu\partial_\mu - m\right)\psi \qquad \textbf{(C. 2)}$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}_1}{\partial(\partial_\mu\bar{\psi})}\right) = -\frac{i\gamma^\mu\partial_\mu\psi}{2}\cdot\exp\left(\frac{i\bar{\psi}\gamma^\mu\partial_\mu\psi - i\left(\partial_\mu\bar{\psi}\right)\gamma^\mu\psi}{2\bar{\psi}A_\mu\gamma^\mu\psi}\right) \qquad \textbf{(C. 4)}$$

(Using $\mathrm{i}\gamma^\mu\partial_\mu\psi = m\psi,\quad -\mathrm{i}\gamma^\mu\partial_\mu\bar{\psi} = m\bar{\psi},\quad \frac{\psi^\dagger\psi}{\bar{\psi}\psi} = \frac{E}{m}$)

$$\left(\frac{\partial\mathcal{L}_1}{\partial\bar{\psi}}\right) - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\bar{\psi}\right)}\right) = 0$$

$$\left[\mathrm{A}_\mu\gamma^\mu\psi - \frac{m^2}{E}\gamma^\mu\psi + m\psi\right]e^{\frac{m^2}{AE}} + \left(\gamma^\mu k_\mu - \mathrm{m}\right)\psi = 0$$

$$\left[\gamma^\mu\left(k_\mu + A_\mu e^{\frac{m^2}{AE}} - \frac{m^2}{E}e^{\frac{m^2}{AE}}\right) - \left(m - me^{\frac{m^2}{AE}}\right)\right]\psi = 0$$

Squaring both terms we will obtain the expression

$$k^2 + 2AEe^{\frac{m^2}{AE}} - 2m^2 e^{\frac{m^2}{AE}} + A^2 e^{\frac{2m^2}{AE}} - \frac{2Am^2}{E} e^{\frac{2m^2}{AE}} + \frac{m^4}{E^2} e^{\frac{2m^2}{AE}}$$

$$= m^2 - 2m^2 e^{\frac{m^2}{AE}} + m^2 e^{\frac{2m^2}{AE}} \quad \textbf{(C. 5)}$$

A is small and $e^{\frac{2m^2}{AE}} \approx 1 + \frac{m^2}{AE}$

$$k^2 = m^2 + \left(1 + \frac{m^2}{AE}\right)\left[-2AE + m^2\left(1 + \frac{m^2}{AE}\right)\right]$$

$$= m^2 - 2AE - 2m^2 + m^2\left(1 + \frac{2m^2}{AE} + \frac{m^4}{A^2E^2}\right)$$

$$= -2AE + \frac{2m^4}{AE} + \frac{m^6}{A^2E^2} \quad \textbf{(C. 6)}$$

Lagrangian 2

$$\mathcal{L}_2 = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi$$

$$+ \frac{\left(i\bar{\psi}\gamma^\mu \partial_\mu \psi - i\left(\partial_\mu \bar{\psi}\right)\gamma^\mu \psi\right)^2}{4\bar{\psi}A_\mu \gamma^\mu \psi} \exp \frac{\left(i\bar{\psi}\gamma^\mu \partial_\mu \psi - i\left(\partial_\mu \bar{\psi}\right)\gamma^\mu \psi\right)^2}{4\left(\bar{\psi}A_\mu \gamma^\mu \psi\right)^2} \quad \textbf{(C. 7)}$$

$$\frac{\partial\mathcal{L}_2}{\partial\bar{\psi}} = \exp\frac{\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi - i(\partial_\mu\bar{\psi})\gamma^\mu\psi\right)^2}{4\left(\bar{\psi}A_\mu\gamma^\mu\psi\right)^2}\left\{\frac{2\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi - i(\partial_\mu\bar{\psi})\gamma^\mu\psi\right)(i\gamma^\mu\partial_\mu\psi)}{4\bar{\psi}A_\mu\gamma^\mu\psi}\right.$$

$$- \frac{\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi - i(\partial_\mu\bar{\psi})\gamma^\mu\psi\right)^2 A_\mu\gamma^\mu\psi}{4\left(\bar{\psi}A_\mu\gamma^\mu\psi\right)^2}$$

$$+ \frac{\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi - i(\partial_\mu\bar{\psi})\gamma^\mu\psi\right)^2}{4\bar{\psi}A_\mu\gamma^\mu\psi}$$

$$\cdot\left[\frac{2\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi - i(\partial_\mu\bar{\psi})\gamma^\mu\psi\right)(i\gamma^\mu\partial_\mu\psi)}{4\left(\bar{\psi}A_\mu\gamma^\mu\psi\right)^2}\right.$$

$$\left.\left. - \frac{2\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi - i(\partial_\mu\bar{\psi})\gamma^\mu\psi\right)^2 A_\mu\gamma^\mu\psi}{4\left(\bar{\psi}A_\mu\gamma^\mu\psi\right)^3}\right]\right\} + (i\gamma^\mu\partial_\mu - m)\psi \qquad \textbf{(C. 8)}$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}_2}{\partial(\partial_\mu\bar{\psi})}\right)$$

$$= e^{\frac{\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi - i(\partial_\mu\bar{\psi})\gamma^\mu\psi\right)^2}{4\left(\bar{\psi}A_\mu\gamma^\mu\psi\right)^2}}\left\{\frac{2\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi - i(\partial_\mu\bar{\psi})\gamma^\mu\psi\right)(-i\gamma^\mu\partial_\mu\psi)}{4\bar{\psi}A_\mu\gamma^\mu\psi}\right.$$

$$\left. + \frac{\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi - i(\partial_\mu\bar{\psi})\gamma^\mu\psi\right)^2}{4\bar{\psi}A_\mu\gamma^\mu\psi}\left[\frac{2\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi - i(\partial_\mu\bar{\psi})\gamma^\mu\psi\right)(-i\gamma^\mu\partial_\mu\psi)}{4\left(\bar{\psi}A_\mu\gamma^\mu\psi\right)^2}\right]\right\} \qquad \textbf{(C. 9)}$$

(Using $\mathrm{i}\gamma^\mu\partial_\mu\psi = m\psi$, $-\mathrm{i}\gamma^\mu\partial_\mu\bar{\psi} = m\bar{\psi}$, $\frac{\psi^\dagger\psi}{\bar{\psi}\psi} = \frac{E}{m}$)

$$\left(\frac{\partial\mathcal{L}_2}{\partial\bar{\psi}}\right) - \partial_\mu\left(\frac{\partial\mathcal{L}_2}{\partial(\partial_\mu\bar{\psi})}\right) = 0$$

$$\mathrm{e}^{\frac{m^4}{A^2E^2}}\left\{\frac{m^3}{AE} - \frac{m^4\gamma^\mu}{AE^2} + \frac{m^7}{A^3E^3} - \frac{2m^8\gamma^\mu}{A^3E^4} + \frac{m^3}{AE} + \frac{m^7}{A^3E^3}\right\}\psi + (\gamma^\mu k_\mu - m)\psi = 0$$

$$\left[\gamma^{\mu}\left(k_{\mu}-\frac{m^4}{AE^2}\mathrm{e}^{\frac{m^4}{A^2E^2}}-\frac{2m^8}{A^3E^4}\mathrm{e}^{\frac{m^4}{A^2E^2}}\right)-\left(m-\frac{2m^3}{AE}-\frac{2m^7}{A^3E^3}\mathrm{e}^{\frac{m^4}{A^2E^2}}\right)\right]\psi=0$$

Squaring both terms we will obtain the expression and let $1/A$ be small and $\mathrm{e}^{\frac{m^4}{A^2E^2}} \approx 1+\frac{m^4}{AE}$

$$k^2+\frac{2m^4}{AE}\mathrm{e}^{\frac{m^4}{A^2E^2}}+\frac{m^8}{A^2E^4}\mathrm{e}^{\frac{2m^4}{A^2E^2}}+\frac{4m^{12}}{A^4E^6}\mathrm{e}^{\frac{2m^4}{A^2E^2}}+\frac{4m^{16}}{A^6E^8}\mathrm{e}^{\frac{2m^4}{A^2E^2}}$$
$$=m^2+\frac{4m^6}{A^2E^2}\mathrm{e}^{\frac{2m^4}{A^2E^2}}+\frac{8m^{10}}{A^4E^4}\mathrm{e}^{\frac{2m^4}{A^2E^2}}+\frac{4m^{14}}{A^6E^6}\mathrm{e}^{\frac{2m^4}{A^2E^2}}$$

$$k^2=m^2+\frac{2m^4}{AE}\mathrm{e}^{\frac{m^4}{A^2E^2}}\approx m^2+\frac{2m^4}{AE} \qquad \textbf{(C. 10)}$$

Lagrangian 3

$$\mathcal{L}_3=\bar{\psi}(i\gamma^{\mu}\partial_{\mu}-m)\psi$$
$$+\frac{\left(i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi-i(\partial_{\mu}\bar{\psi})\gamma^{\mu}\psi\right)^2}{4\bar{\psi}A_{\mu}\gamma^{\mu}\psi}\exp\left[-\frac{2\bar{\psi}A_{\mu}\gamma^{\mu}\psi}{i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi-i(\partial_{\mu}\bar{\psi})\gamma^{\mu}\psi}\right] \qquad \textbf{(C. 11)}$$

$$\frac{\partial\mathcal{L}_3}{\partial\bar{\psi}}=\mathrm{e}^{-\frac{2\bar{\psi}A_{\mu}\gamma^{\mu}\psi}{i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi-i(\partial_{\mu}\bar{\psi})\gamma^{\mu}\psi}}\left\{\frac{2\left(i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi-i(\partial_{\mu}\bar{\psi})\gamma^{\mu}\psi\right)\left(i\gamma^{\mu}\partial_{\mu}\psi\right)}{4\bar{\psi}A_{\mu}\gamma^{\mu}\psi}\right.$$
$$-\frac{\left(i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi-i(\partial_{\mu}\bar{\psi})\gamma^{\mu}\psi\right)^2\cdot 4A_{\mu}\gamma^{\mu}\psi}{\left(4\bar{\psi}A_{\mu}\gamma^{\mu}\psi\right)^2}$$
$$-\frac{\left(i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi-i(\partial_{\mu}\bar{\psi})\gamma^{\mu}\psi\right)^2}{4\bar{\psi}A_{\mu}\gamma^{\mu}\psi}$$
$$\left.\cdot\left[\frac{2A_{\mu}\gamma^{\mu}\psi}{i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi-i(\partial_{\mu}\bar{\psi})\gamma^{\mu}\psi}-\frac{2\bar{\psi}A_{\mu}\gamma^{\mu}\psi\cdot i\gamma^{\mu}\partial_{\mu}\psi}{\left(i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi-i(\partial_{\mu}\bar{\psi})\gamma^{\mu}\psi\right)^2}\right]\right\} \qquad \textbf{(C. 12)}$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}_2}{\partial(\partial_\mu\bar{\psi})}\right)=\partial_\mu\left\{\mathrm{e}^{-\frac{2\bar{\psi}A_\mu\gamma^\mu\psi}{i\bar{\psi}\gamma^\mu\partial_\mu\psi-i(\partial_\mu\bar{\psi})\gamma^\mu\psi}}\left[\frac{2\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi-i\partial_\mu\bar{\psi}\gamma^\mu\psi\right)\left(-i\gamma^\mu\partial_\mu\psi\right.}{4\bar{\psi}A_\mu\gamma^\mu\psi}\right.\right.$$

$$\left.\left.+\frac{\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi-i\partial_\mu\bar{\psi}\gamma^\mu\psi\right)^2}{4\bar{\psi}A_\mu\gamma^\mu\psi}\cdot\left(\frac{2\bar{\psi}A_\mu\gamma^\mu\psi\cdot(-i\gamma^\mu\partial_\mu\psi)}{\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi-i\partial_\mu\bar{\psi}\gamma^\mu\psi\right)^2}\right)\right]\right\} \qquad \textbf{(C. 13)}$$

(Using $\mathrm{i}\gamma^\mu\partial_\mu\psi=m\psi,\ \ -\mathrm{i}\gamma^\mu\partial_\mu\bar{\psi}=m\bar{\psi},\ \ \frac{\psi^\dagger\psi}{\bar{\psi}\psi}=\frac{E}{m}$)

$$\left(\frac{\partial\mathcal{L}_2}{\partial\bar{\psi}}\right)-\partial_\mu\left(\frac{\partial\mathcal{L}_2}{\partial(\partial_\mu\bar{\psi})}\right)=0$$

$$\left(\gamma^\mu k_\mu-m\right)\psi+\mathrm{e}^{\frac{AE}{m^2}}\left\{\begin{matrix}\frac{m^2\bar{\psi}\psi\psi}{A_\mu\psi^\dagger\psi}-\frac{(m\bar{\psi}\psi)^2A_\mu\gamma^\mu\psi}{\left(A_\mu\psi^\dagger\psi\right)^2}+\frac{m^2\bar{\psi}\psi}{2A_\mu\psi^\dagger\psi}\cdot m\psi\\ -\frac{(m\bar{\psi}\psi)^2}{A_\mu\psi^\dagger\psi}\left[\frac{A_\mu\gamma^\mu\psi}{m\bar{\psi}\psi}-\frac{2A_\mu\psi^\dagger\psi m\psi}{(2m\bar{\psi}\psi)^2}\right]+\frac{m\psi}{2}\end{matrix}\right\}=0$$

$$\left[\gamma^\mu\left(k_\mu-\mathrm{e}^{\frac{AE}{m^2}}\frac{m^4}{AE^2}-\mathrm{e}^{\frac{AE}{m^2}}\frac{m^2}{E}\right)-\left(m-\mathrm{e}^{\frac{AE}{m^2}}\frac{2m^3}{AE}-\mathrm{e}^{\frac{AE}{m^2}}m\right)\right]\psi=0$$

Squaring both terms we will obtain the expression and let $1/A$ be small.

$$k^2-\frac{2m^4}{AE}\mathrm{e}^{\frac{AE}{m^2}}-2m^2\mathrm{e}^{\frac{AE}{m^2}}+\frac{m^8}{A^2E^4}\mathrm{e}^{\frac{2AE}{m^2}}+\frac{2m^6}{AE^3}\mathrm{e}^{\frac{2AE}{m^2}}+\frac{m^4}{E^2}\mathrm{e}^{\frac{2AE}{m^2}}$$

$$=m^2-\frac{4m^4}{AE}\mathrm{e}^{\frac{AE}{m^2}}-2m^2\mathrm{e}^{\frac{AE}{m^2}}+\frac{4m^6}{A^2E^2}\mathrm{e}^{\frac{2AE}{m^2}}+\frac{4m^4}{AE}\mathrm{e}^{\frac{2AE}{m^2}}+m^2\mathrm{e}^{\frac{2AE}{m^2}}$$

$$k^2=m^2+m^2\mathrm{e}^{\frac{2AE}{m^2}} \qquad \textbf{(C. 14)}$$

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