Exploring atom-photon interactions in one dimensional waveguide and its applications

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Abstract

There have been many theoretical studies on atom photon interaction in a waveguide which uses quantum trajectory method or master equations which are intrinsically non-deterministic. By using real-space Hamilitonian and Bethe-Ansatz, we can derive the deterministic transport properties of these interactions.

In this paper, we will derive the transport properties of photon in waveguide. By finding out the response function of the atom-photon interaction, we show that we can understand the transport properties of mirrors surrounding an atoms and N atoms in a waveguide using the transfer matrix method. We can also build similar systems with with different kind of atoms(3-level atoms), once we know the transmission and reflection coefficient. Due to experimental advancement for atom-photon interactions in a waveguide, it is a rich field which can produce scalable quantum devices and quantum network communications.

Contents

Chapter 1

Introduction

Controllable atom-photon interaction is of central importance in quantum information processing. There had been many theoretical works done regarding the photon transport properties of a wavelength-scale cavity coupled to a two or multi-level system. Such systems are usually studied with quantum trajectory method,[8, 10, 28] with a Monte Carlo approach which is non-deterministic in nature.[20] Other approaches includes deriving a master equation[14], which assumes a weak input coherent state and it has uncontrolled approximations[29]. In this paper we provide a full quantum mechanical approach to solve the transport properties of photons in a waveguide interacting with atoms.

Figure 1.1: An atom in an one dimensional waveguide

Motivations for studying waveguide-based atom photon interactions are due to huge leaps in advancement in experimental progress. There are experimental systems include a metallic nanowire coupled to a quantum dot[1], which is a nanocrystal made of semiconductor materials that are small enough to exhibit quantum mechanical properties, a diamond nanowire coupled to a quantum dot,[4] cold atoms trapped inside a hollow fibre,[5] a one dimensional superconducting nanowire coupled to

a qubit[6] and a Gallium Arsenic photonic nanowire with embedded InAs quantum dots[6]. Even though the atom are side coupled to the waveguide in some of the experiment instead of being embedded in the waveguide mode, it has been experimentally demonstrated that 90% of the spontaneously emitted photon has been guided back into the waveguide. This phenomenon happens in the strong-coupling regime[6].

In this paper, we are going to consider 2-level and 3-level systems coupled to a one dimensional waveguide. We transform Dicke's Hamiltonian from frequency domain to real space. Using Bethe-Ansatz, the transport properties for a single atom-photon interaction can be derived. We then formulate the transfer matrix for that atom to find bandgap structures for a linear chain of atom and an asymmetrical lineshape for transmission spectrum when 2 mirrors surround the atom. The two photon scattering matrix is formulate from Bethe-Ansatz and doing a completeness check of the two-photon Hilbert space first derived from the ansatz.

The paper is organised as follows, in chapter 1 to 5, we are going to derive the transport properties of photon and 2 level systems. Starting with single atom and photon, the transfer matrix of the single atom is built up. Subsequently, we are going to use the transfer matrix method to describe infinite and finite chains of atoms in the waveguide. In chapter 6, we derive the transmission and reflection spectra of a driven Λ atom and in chapter 7, we derive scattering matrix for 2 photons. In chapter 8, future applications of this waveguide based systems will be discussed followed by the conclusion.

Chapter 2

Transmission and reflection coefficients

In this chapter we will begin solving for the transmission and reflection coefficients for a single photon incident on an atom in a waveguide. Transmission and reflection coefficients can tell us the transport properties of the atom. In the transfer matrix method, they can be used to form composite systems in the waveguide which will be discussed in later chapters.

2.1 Hamiltonian of the system

The interaction between photon and the 2-level atoms is described by Dicke Hamiltonian[12]

$$H = \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}} + \frac{1}{2}\hbar\Omega S_z + \sum_{\mathbf{k}} V_{\mathbf{k}}(a^{\dagger}_{\mathbf{k}} + a_{\mathbf{k}})(S_+ + S_-) \tag{2.1}$$

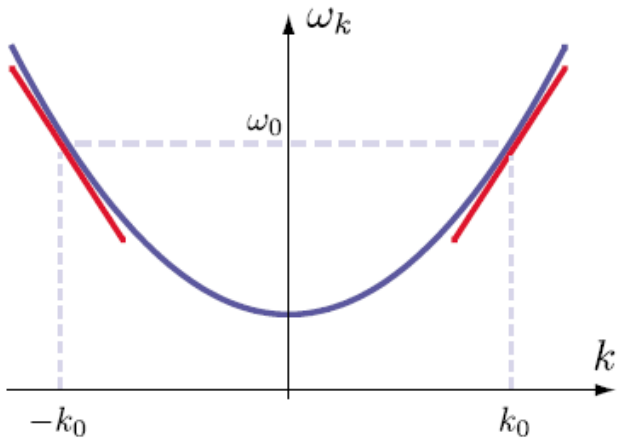

Figure 2.1: Dispersion relation of a one dimensional waveguide, which can be approximated to be a linear function in the vicinity of ω_0.[27]

To rewrite Dicke's Hamiltonian in real space, we linearise the dispersion relation of the one dimensional waveguide. The typical dispersion relation of a one dimensional waveguide is a quadratic function. For any frequency, ω_0 away from the cut off frequency, we can approximate the dispersion relation to,

$$\omega_{\mathbf{k_R}} \approx \omega_{k=k_0} \approx v_g(k-k_0)+\omega_0 \equiv v_g k_R + \omega_0 \tag{2.2}$$

$$\omega_{\mathbf{k_L}} \approx \omega_{k=-k_0} \approx -v_g(k+k_0)+\omega_0 \equiv -v_g k_L + \omega_0 \tag{2.3}$$

with the corresponding wave number, $\pm k_0$ and the subscripts R and L are labels for the left and right branch. After linearising the dispersion relation, we have

$$\sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \approx \sum_{\mathbf{k_R}} \hbar\omega_{\mathbf{k_R}} a^\dagger_{\mathbf{k_R}} a_{\mathbf{k_R}} + \sum_{\mathbf{k_L}} \hbar\omega_{\mathbf{k_L}} a^\dagger_{\mathbf{k_L}} a_{\mathbf{k_L}} \tag{2.4}$$

To represent Eq. (2.4) in real space, we have to define the Fourier transform,

$$a_{k_R} = \int \mathrm{d}x\, c_R(x) e^{-ik_R x} \tag{2.5}$$

$$a^\dagger_{k_R} = \int \mathrm{d}x\, c^\dagger_R(x) e^{+ik_R x} \tag{2.6}$$

where $c^\dagger_R(x)$ creates a right moving photon and $c_R(x)$ annihilate a right moving photon. The first term of Eq. (2.4) becomes

$$\begin{aligned}
\sum_{\mathbf{k_R}} \omega_{\mathbf{k_R}} a^\dagger_{\mathbf{k_R}} a_{\mathbf{k_R}} &= \sum_{\mathbf{k_R}} \omega_{\mathbf{k_R}} \int \mathrm{d}x\mathrm{d}x'\, c^\dagger_R(x) c_R(x') e^{ik_R(x-x')} \\
&= \int \mathrm{d}x\mathrm{d}x'\, c^\dagger_R(x) c_R(x') \int \frac{\mathrm{d}k_R}{2\pi}\, \omega_{k_R} e^{ik_R(x-x')} \\
&= \int \mathrm{d}x\mathrm{d}x'\, c^\dagger_R(x) c_R(x') \left(\omega_0 - iv_g \frac{\partial}{\partial x}\right) \int \frac{\mathrm{d}k_R}{2\pi}\, e^{ik_R(x-x')} \\
&= \int \mathrm{d}x\mathrm{d}x'\, c^\dagger_R(x) c_R(x') \left(\omega_0 + iv_g \frac{\partial}{\partial x}\right) \delta(x-x') \\
&= \int \mathrm{d}x\, c^\dagger_R(x) \left(\omega_0 - iv_g \frac{\partial}{\partial x}\right) c_R(x)
\end{aligned} \tag{2.7}$$

Similarly we can obtain the real space representation for a left moving photon by applying the same treatment,

$$\sum_{\mathbf{k_L}} \omega_{\mathbf{k_L}} a^\dagger_{\mathbf{k_L}} a_{\mathbf{k_L}} = \int \mathrm{d}x\, c^\dagger_L(x) \left(\omega_0 + iv_g \frac{\partial}{\partial x}\right) c_L(x) \tag{2.8}$$

The interaction term can be simplified by rotating wave approximation where only the cross term is kept. This is a valid approximation when the applied electromagnetic radiation is near resonance with an atomic transition.

$$\begin{aligned}
\sum_{\mathbf{k}} V_{\mathbf{k}} a^\dagger_{\mathbf{k}} S_- + a_{\mathbf{k}} S_+ &= \sum_{\mathbf{k_L}} a^\dagger_{\mathbf{k_L}} S_- + a_{\mathbf{k_L}} S_+ + \sum_{\mathbf{k_R}} a^\dagger_{\mathbf{k_R}} S_- + a_{\mathbf{k_R}} S_+ \\
&= \int \mathrm{d}x\, \delta(x) \left[c^\dagger_R(x) S_- + c^\dagger_L(x) S_- + c_R(x) S_+ + c_L(x) S_+\right]
\end{aligned} \tag{2.9}$$

Bringing all the terms together, we obtain the final Hamiltonian in real space

$$
\begin{aligned}
H = \int & -iv_g c_R^\dagger(x)\frac{\partial}{\partial x}c_R(x) + iv_g c_L^\dagger(x)\frac{\partial}{\partial x}c_L(x) \\
& + V\delta(x)\left[c_R^\dagger(x)S_- + c_R(x)S_+ + c_L^\dagger(x)S_- + c_L(x)S_+\right]\mathrm{d}x \\
& + E_e S_+ S_- + E_g S_- S_+
\end{aligned} \tag{2.10}
$$

2.2 Solutions to the Hamiltonian

Assuming a photon is coming from the left with energy $E_k = v_g k$, the stationary state of the system can be written as,

$$
|E_k\rangle = \int \mathrm{d}x \left[\phi_{k,R}(x)c_R^\dagger(x) + \phi_{k,L}(x)c_L^\dagger(x)\right]|0,g\rangle + e_k a_e^\dagger a_g|0,g\rangle \tag{2.11}
$$

Where e_k is the probability for the atom to be in the excited state. For a photon incident from the left, $\phi_{k,R}$ and $\phi_{k,L}$ take the form,

$$
\phi_{k,R} = e^{ikx}\theta(-x) + te^{ikx}\theta(x) \tag{2.12}
$$

$$
\phi_{k,L} = re^{-ikx}\theta(-x) \tag{2.13}
$$

where t and r are the transmission and reflection amplitude, respectively. From the eigenvalue equation and the commutator relation $[c_{L/R}(x), c_{L/R}^\dagger(x')] = \delta(x-x')$, we can obtain the expression of t and r.

$$
\begin{aligned}
-iv_g\frac{\partial}{\partial x}\phi_{k,R}(x) + e_k V\delta(x) + E_g\phi_{k,R}(x) &= E_k\phi_{k,R}(x) \\
iv_g\frac{\partial}{\partial x}\phi_{k,L}(x) + e_k V\delta(x) + E_g\phi_{k,L}(x) &= E_k\phi_{k,L}(x) \\
V[\phi_{k,R}(0) + \phi_{k,L}(0)] + E_e e_k &= E_k e_k
\end{aligned}
$$

When defined as a piecewise constant function, the Heaviside step function is given by

$$\theta(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases} \tag{2.14}$$

The equations of motion can be written in matrix form after evaluation. The coefficients t, r and e_k can be easily solved using Gauss-Jordan elimination.

$$\begin{pmatrix} -iv_g & 0 & V \\ 0 & -iv_g & V \\ \frac{V}{2} & \frac{V}{2} & \Omega - E_k \end{pmatrix} \begin{pmatrix} t \\ r \\ e_k \end{pmatrix} = \begin{pmatrix} iv_g \\ 0 \\ \frac{V}{2} \end{pmatrix} \tag{2.15}$$

We obtain the expression for the probability amplitude, e_k

$$\begin{aligned} e_k &= \frac{V}{i\frac{V^2}{v_g} - (\Omega - E_k)} \\ &= \frac{-V}{\left(\frac{V^2}{v_g}\right)^2 + (\Omega - E_k)^2} \left(i\frac{V^2}{v_g} + (\Omega - E_k) \right) \\ &= -\frac{v_g}{V} \sin(b) e^{ib}, \quad b = \arctan\left(\frac{V^2}{v_g(\Omega - E_k)} \right) \end{aligned} \tag{2.16}$$

By defining the phase difference, b, we can express the transmission and reflection coefficient in terms of b.

$$t = \cos b e^{ib}, \quad r = i \sin b e^{ib}, \quad e_k = -\frac{v_g}{V} \sin b e^{ib} \tag{2.17}$$

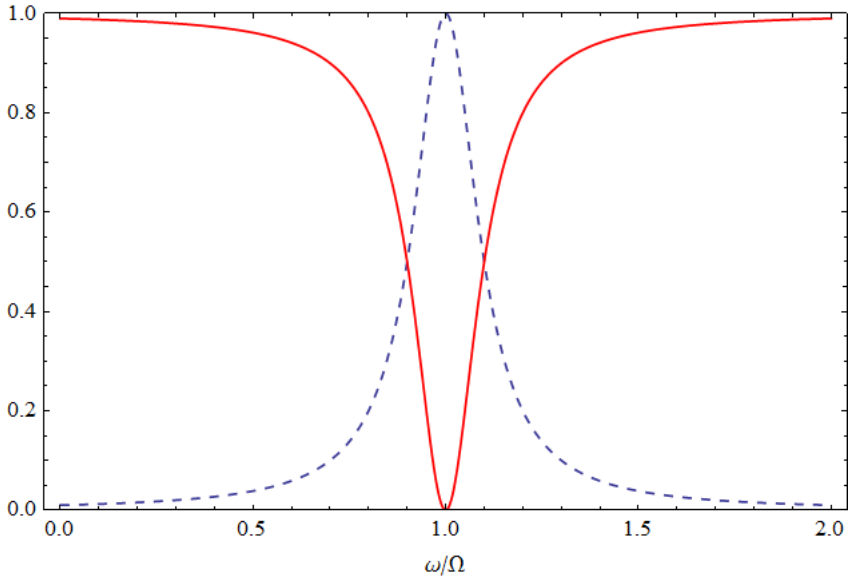

Figure 2.2: Transmission spectrum (solid line) and the reflection spectrum (dashed line) for an atom in a one dimensional waveguide. $V^2/v_g = 0.1\Omega$

We have shown that in a one dimensional waveguide, when a single photon is incident upon the two-level system with a frequency on resonance, it results in the photon being completely reflected with no loss. The wave function of the spontaneously emitted photon will definitely interfere with the incident wave with no loss, as there are only backward and forward directions in the waveguide.[25]

Chapter 3

Fano lineshape

A Fano resonance is a type of resonant scattering phenomenon that gives rise to an asymmetric line-shape.[13] Interference between a background and a resonant scattering process produces the asymmetric line-shape, the photon modes are not purely in the forward and backward direction.[25] We can achieve this by surrounding the atom with a pair of partially reflecting dielectric mirrors as shown in Fig. (3.1).

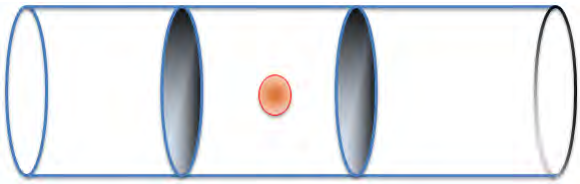

Figure 3.1: An atom surrounded by two partially reflecting dielectric slab

To generate the Fano line shape, we need to cascade the transfer matrix of the atom, the partially reflecting dielectric slab and propagation of the photon in free space.

For the partially reflecting dielectric slab, the transfer matrix can be described as[25]

$$M_{slab} = \frac{1}{i\sqrt{1-r^2}} \begin{pmatrix} -1 & -r \\ r & 1 \end{pmatrix} \tag{3.1}$$

where r is the amplitude reflectivity of the slab. For the transfer matrix of the atom, we can make use of the transmission and reflection coefficients derived in Eq. (2.17) and substitute it in the

transfer matrix of an atomic plane

$$M_{atom} = \begin{pmatrix} 1/t^* & -r^*/t^* \\ -r/t & 1/t \end{pmatrix} = \begin{pmatrix} 1 - \dfrac{iV^2}{v_g(\Omega - E_k)} & -\dfrac{iV^2}{v_g(\Omega - E_k)} \\ \dfrac{iV^2}{v_g(\Omega - E_k)} & 1 + \dfrac{iV^2}{v_g(\Omega - E_k)} \end{pmatrix} \tag{3.2}$$

For the transfer matrix for free propagation, we can describe it as

$$M_{prop} = \begin{pmatrix} e^{ikd} & 0 \\ 0 & e^{-ikd} \end{pmatrix} \tag{3.3}$$

where d is the distance between the atom and the mirror.

3.1 Response function of mirrors surrounding an atom

We can find the response function of the system by cascading the transfer matrices above.[11] The order of the matrices is determined by arrangement of the mirrors and atom, If the system if not symmetrical, we need to trace the path of the incoming photon.

$$\mathrm{M_{system}} = \mathrm{M_{slab}M_{prop}M_{atom}M_{prop}M_{slab}} \tag{3.4}$$

For any transfer matrix, the reflection and the transmission coefficients associated with the entire element are

$$r = -\frac{M_{21}}{M_{22}}, \quad t = \frac{1}{M_{22}} \tag{3.5}$$

From Eq. (3.4) and Eq. (3.4), the transmission coefficient is

$$t = \frac{1 - r^2}{e^{2ikd}r^2(1 - iz) + e^{-2ikd}(1 + iz) - 2irz} \tag{3.6}$$

where $z = \dfrac{V^2}{v_g(\Omega - E_k)}$. The transimission spectrum is obtained by squaring the transmission coefficient.

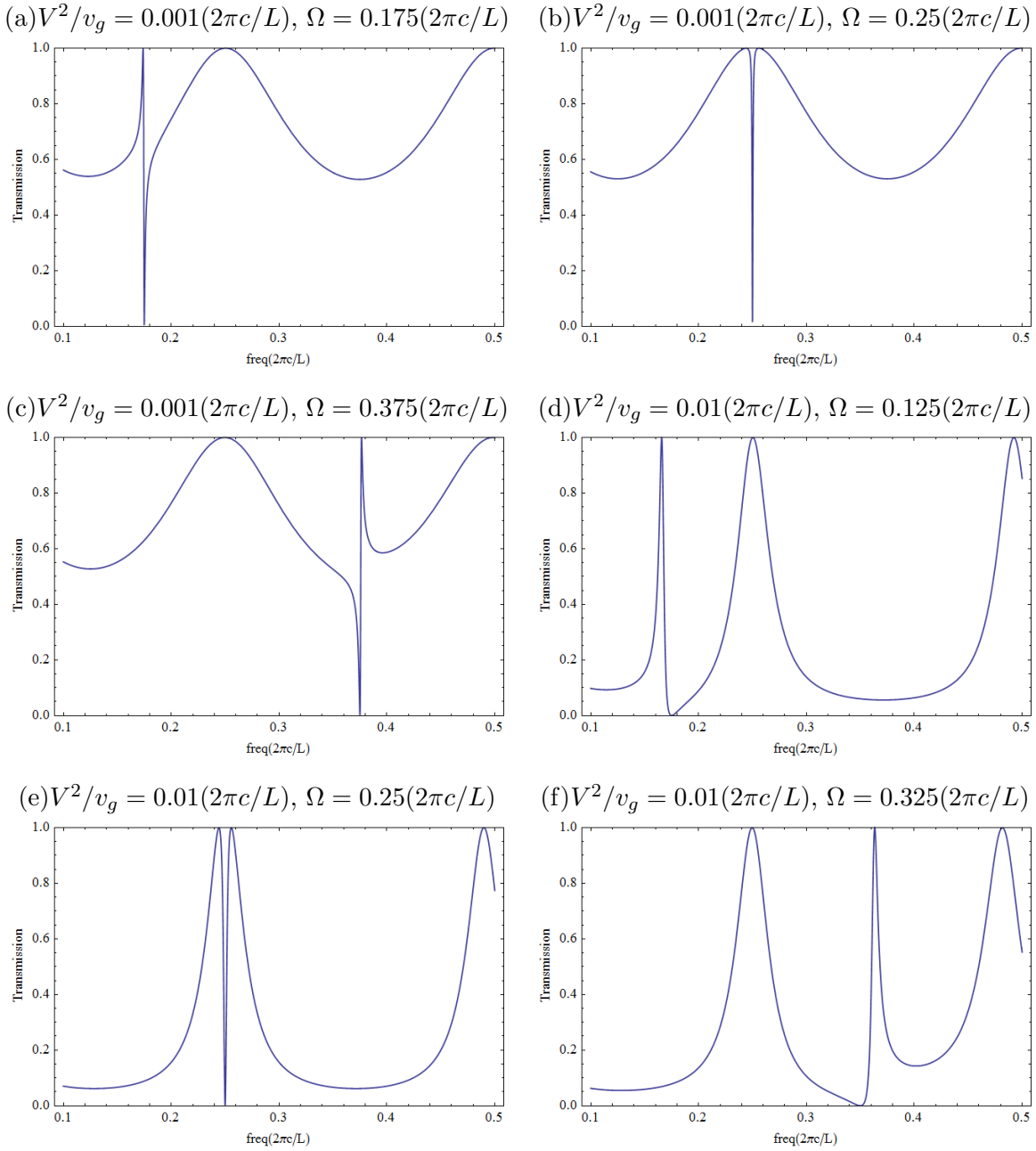

Figure 3.2: Transmission spectra through the optical system of mirrors and atom. (a)-(c) $r = 0.4$ (d)-(f) $r = 0.8$

Without the 2-level atom in the waveguide, Fig. (3.2) will only consist of Fabry-Perot oscillations,

Fig. (3.3).

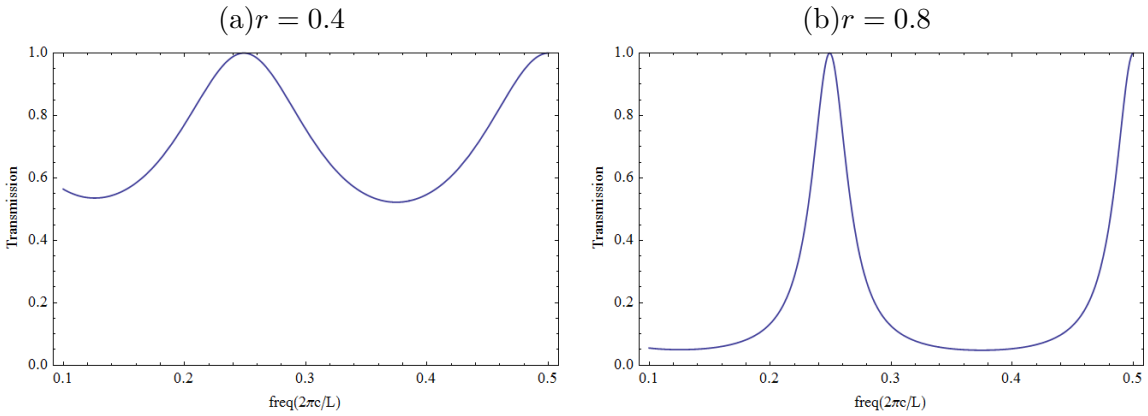

Figure 3.3: Transmission spectra through the optical system of mirrors only.

We have shown that in smaller r(≤ 0.4), the transmission spectra will consist of a Fano lineshape occurring at the resonance of the atom. It is superimposed on the Fabry-Perot oscillation with its width proportional to V^2/v_g. For larger r(≥ 0.8), the transmission spectrum shows tunnelling peaks.

Chapter 4

Linear chain of atoms

By cascading the transfer matrices, we can study the transport properties of a chain of any length of resonant atoms, periodic or disordered. To implement a chain of periodic atoms, it has been proposed that we can use fiber-based atom trap, using the evanescent wave of the propagating photon as the trapping potential.[22]

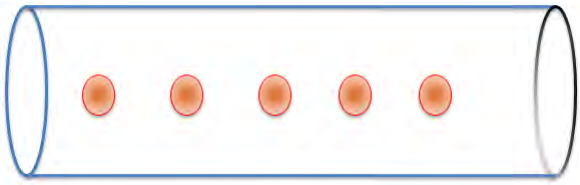

Figure 4.1: A chain of atoms.

The transfer matrix for an individual cell consisting of the atom plane and free space is given by Eq. (3.2) and Eq. (3.3),

$$\begin{aligned} Q &= \mathrm{M_{prop}M_{atom}} \\ &= \begin{pmatrix} e^{ikd}\left(1-\dfrac{iV^2}{v_g(\Omega-E_k)}\right) & -e^{ikd}\dfrac{iV^2}{v_g(\Omega-E_k)} \\ e^{ikd}\dfrac{iV^2}{v_g(\Omega-E_k)} & e^{-ikd}\left(1+\dfrac{iV^2}{v_g(\Omega-E_k)}\right) \end{pmatrix} \end{aligned} \tag{4.1}$$

In this chapter, we look at the dispersion relation of an infinite chain of atom by applying Bloch's theorem.

4.1 Bloch's Theorem

According to Bloch's theorem, the eigenfunctions of the wave equations for a periodic potential are also periodic in nature.[17] The wavefunctions described by Bloch theorem is most commonly an electron in a crystal lattice. It has been shown that analogous bandgap exist when photons propagate through a periodic potential wells after studies by Yablonovitch and Gmitter.[33]

The dispersion relation can be obtained from Bloch's theorem which uses the periodic boundary condition to solve for transmission and reflection coefficients.

$$\begin{pmatrix} a \\ b \end{pmatrix} = e^{-i\mathbf{K}d} \begin{pmatrix} a' \\ b' \end{pmatrix} \tag{4.2}$$

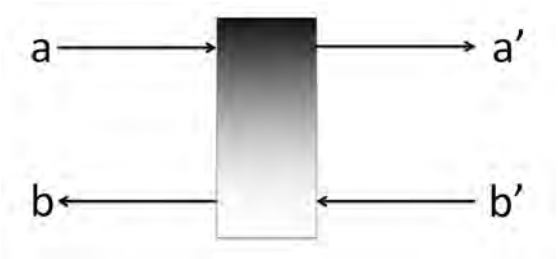

Figure 4.2: a and b are incoming and outgoing wave amplitudes on one side of the atom while a' and b' are the outgoing and incoming wave amplitudes on the other side of the atom.

By equating the transfer matrix, Q which a single cell of atom and free space, and Eq. (4.2), we obtain the dispersion relation for the system of infinite chain of atom.

$$\cos(\mathbf{K}d) = \frac{1}{2}Tr(Q) \tag{4.3}$$

where $\mathbf{K}$ is the Bloch wave vector. When we evaluate the trace of the transfer matrix, we obtain the dispersion relation,

$$\cos(\mathbf{K}d) = \cos(kd) - \frac{V^2}{v_g(\Omega - E_k)}\sin(kd) \tag{4.4}$$

4.2 Dispersion relation

To plot the dispersion relation between frequency ω and Bloch wave vector $\mathbf{K}$, The terms in Eq. (4.4) are shifted the to left hand side. The plot is obtained by plotting a multi-variable function with ω and $\mathbf{K}$. As the wave number, kd represents the momentum of the incoming photon, we can transform the variable,

$$\begin{aligned} kd &= \frac{\omega}{v_g}\left(\frac{d}{\lambda_r}\right)\lambda_r \\ &= \frac{\omega}{v_g}\left(\frac{d}{\lambda_r}\right)\frac{2\pi v_g}{\Omega} \\ &= 2\pi\frac{\omega}{\Omega}\left(\frac{d}{\lambda_r}\right) \end{aligned} \tag{4.5}$$

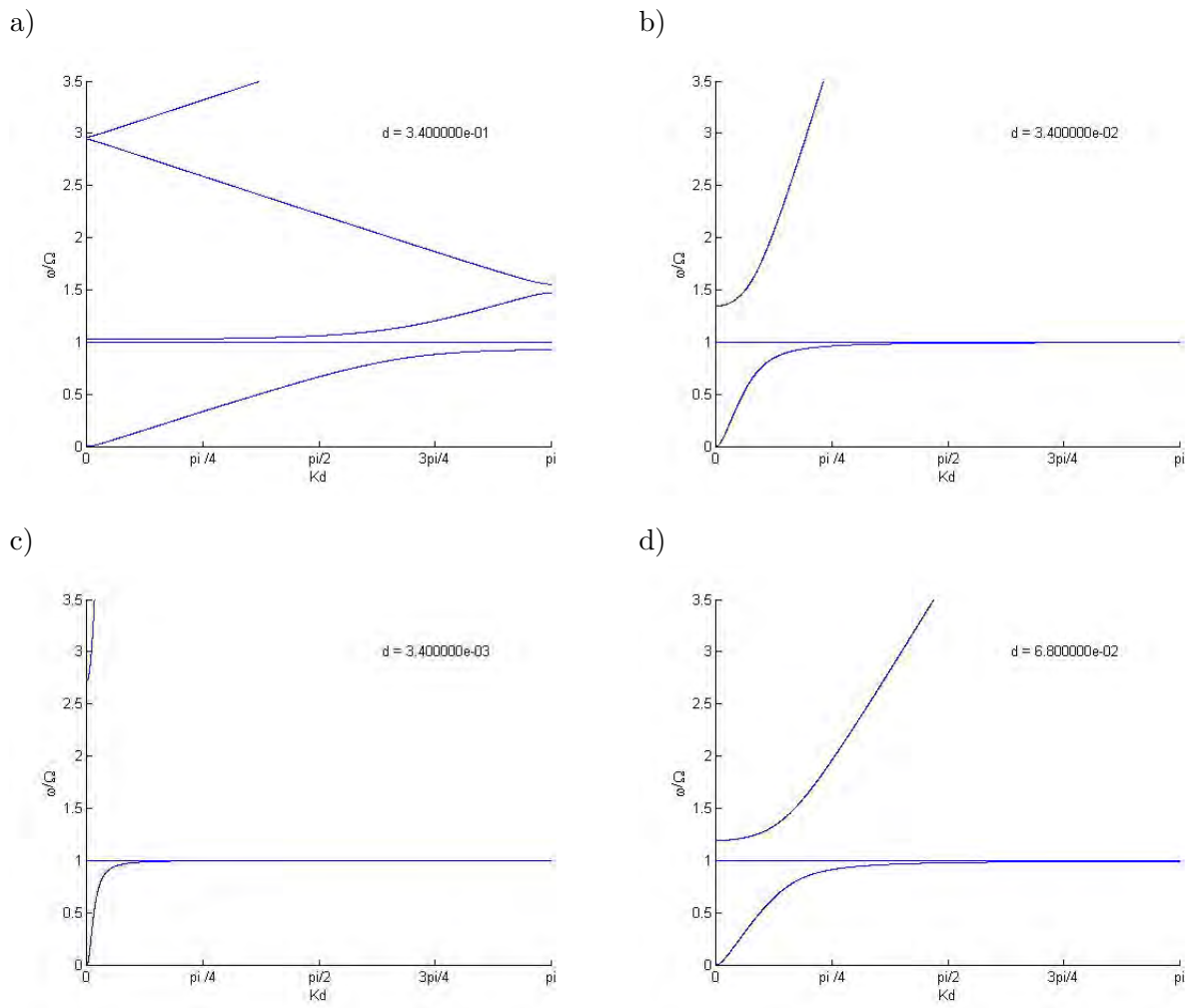

Figure 4.3: Dispersion relation between frequency ω and Bloch wave vector $\mathbf{K}.\lambda_r = 2\pi v_g/\Omega$ and $V^2/v_g\Omega = 0.05$. a) $d/\lambda_r = 3.4\times10^{-3}$ b) $d/\lambda_r = 3.4\times10^{-2}$ c)$d/\lambda_r = 6.8\times10^{-2}$ d)$d/\lambda_r = 3.4\times10^{-1}$

From Fig. (4.3), we can see that the bandgap can be tuned by adjusting the distance between the atoms. The bandgap is large is when the atoms are close to each other with respect to the resonance wavelength, which is in contrast to a small bandgap when they are further apart. This can be explained by the constructive interference of the photon waves to form a large bandgap when the the atom are close enough to be considered to be continuous with a linear density distribution.

Chapter 5

N atoms in a waveguide

In the previous chapter, we discuss the effects of having a linear chain of atom. However, the calculation done was for a periodic boundary condition which correspond to an infinite number of atom. From Eq. (4.1), The transfer matrix of the system can be obtained and the transmission and reflection function can be evaluated.

5.1 Constant periodicity

In a constant periodic system, the reflection spectra is shown in Fig. (5.1). By cascading the transfer matrix of a single cell, Eq. (4.1), the transfer matrix of the system is obtained. From Eq. (3.5), the transmission and reflection coefficient can be obtained. Fig. (5.1) shows that we can control the photon transport by varying the coupling constant and the number of atoms.

There is total reflection in the region of resonance energy and with the width proportional to the coupling coefficient. The result is consistent with Fig. (4.3) where the bandgap occurs around resonance frequency of the 2-level atom but we can also see bandgaps occuring at frequecies away from the resonance.

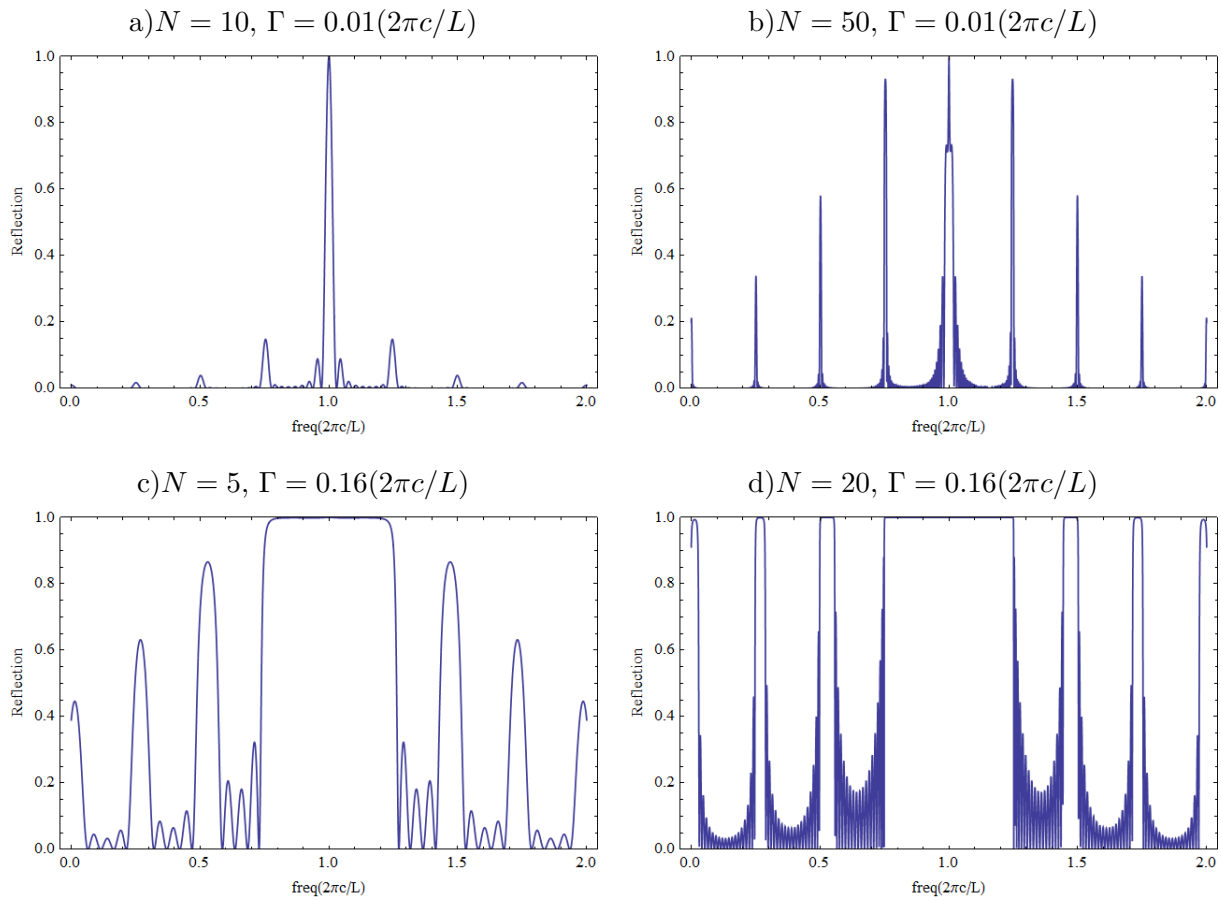

Figure 5.1: Reflection spectra of a system consisting a chain of atom with constant periodicity.

5.2 Randomised periodicity

In the previous section, we considered the photon transport through a waveguide coupled to perfectly periodic 2-level atoms. We randomise a periodic system by generating a list of random periodicities, d_n following a normal distribution. After which we substitute into the transfer matrix of each individual cell, Eq. (4.1). The transfer matrix of the system had been computed by varying the number of atom over 100 realisation.

The spectra obtained is not similar to Fig. (5.1). There is no bands of allowed transmission like in the previous plot but only a single bandgap. It is similar to a phenomenon where an electron moving through a disordered lattice is trapped, which is known as Anderson localisation.[2]

Anderson localization originated in wave interference between multiple-scattering paths of electrons. In the strong scattering limit, the severe interferences can completely halt the waves inside the disordered medium. As a result, the material is transformed from a conductor to an insulator. Although the phenomenon had been predicted in the context of electrons, it has been experimentally observed in many different media. Experiments had shown that a non-interacting Bose-Einstein condensate in a one-dimensional quasi-periodic lattice demonstrates localization[21] and photons transport can also be localised in an disordered system[31, 23].

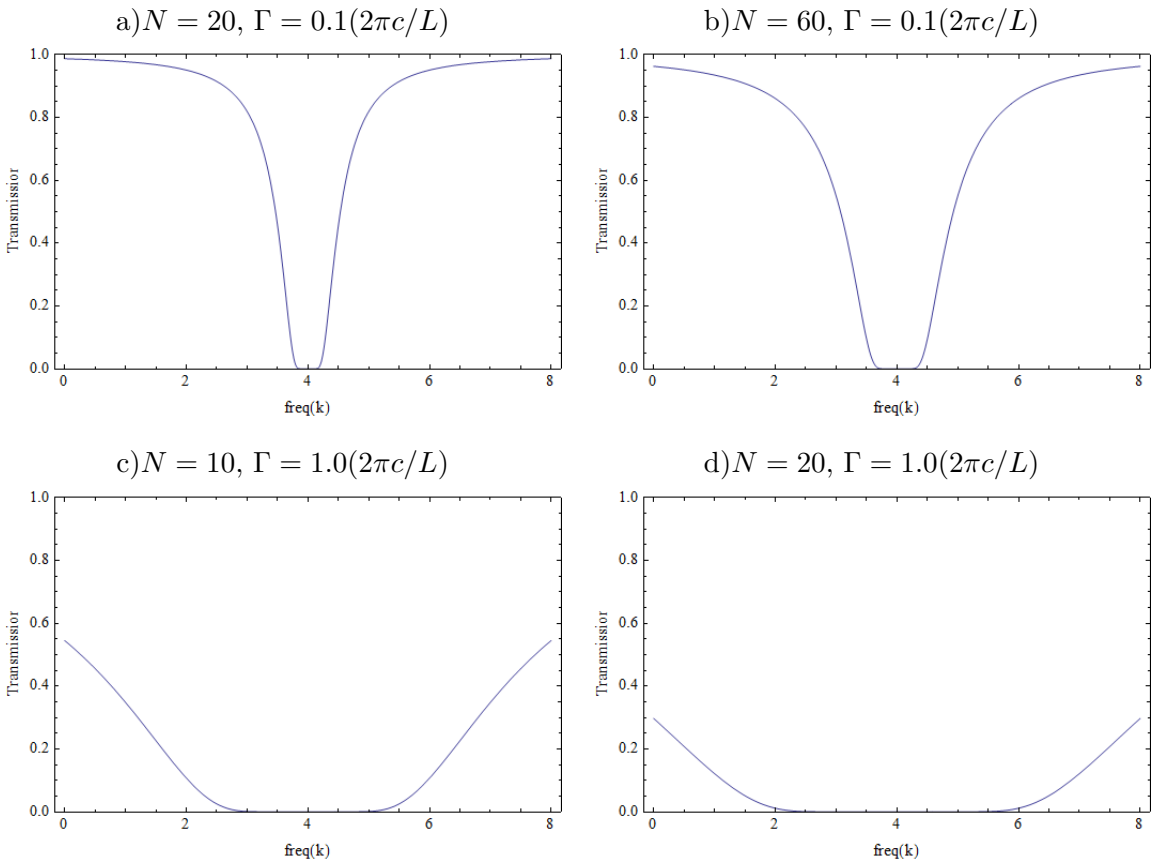

Figure 5.2: Transmission spectra of a system consisting a chain of atom with randomised periodicity.

By plotting localisation length against the frequency of the incoming photon, we can determine quantitatively the localisation of the photon. Localisation length, ϑ is the property of the transport material, which in our case is a single photon and it characterises the spatial extend of its wave-function. It can be found by equating the eigenvalues of the system's transfer matrix and equating them to $e^{Nd_0/\vartheta}$, where d_0 is the mean periodicity of the system. The transfer matrix of the system

has been computed using $N = 100$ randomly chosen values for periodicity and each localisation length over the spectrum has been averaged over 100 simulations.

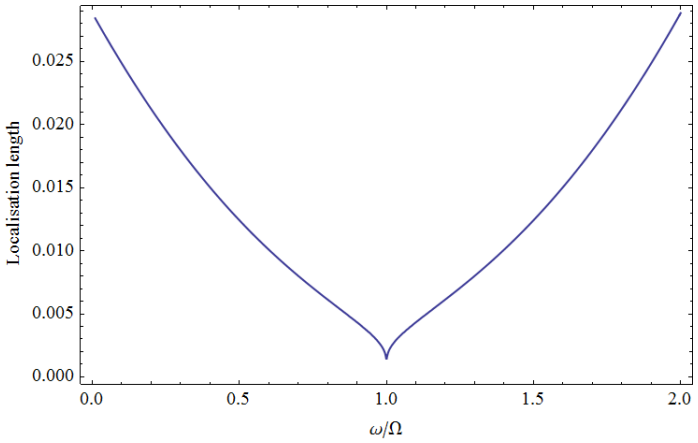

Figure 5.3: Anderson localisation in a waveguide coupled to random array of atoms.$\Gamma = 1.5\Omega$, $d_0 = 2\pi c/\omega$

From Fig. (5.3), we can observes that the localisation length of the of the photon tends to zero at resonance and we have shown that Anderson localisation can be achieved through strong coupling and random arrangements of the atom in the waveguide.

Chapter 6

Three level atom

In this section, we solve the scattering problem for a single photon and a driven three-level emitter with three internal states.[32] Electromagnetically induced transparency (EIT) is a nonlinear property of a medium which render medium transparent over a narrow spectral range. It is usually accomplished in three-level atom where two coherent excitation probabilities interfere destructively leading to a dark state.[24]

6.1 Hamiltionian of a driven Λ-type atom

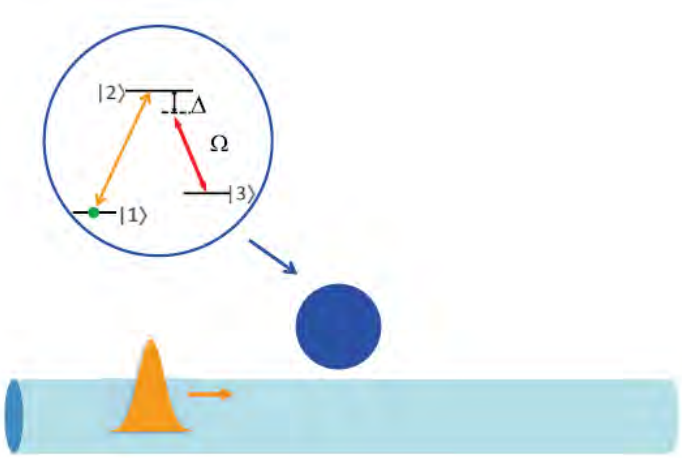

Figure 6.1: A driven Λ-type atom.

We start with a driven Λ atom where the excited atomic state $|2\rangle$ is coupled to another level $|3\rangle$ by a coupling field with Rabi frequency Ω and detuning Δ. By applying rotating wave approximation and setting the ground state to zero, the atomic Hamiltonian is given by

$$H_{atom} = (E_2 - i\gamma_2/2)|2\rangle\langle 2| + (E_2 - \Delta - i\gamma_3/2)|3\rangle\langle 3| + \frac{\Omega}{2}(|3\rangle\langle 2| + |2\rangle\langle 3|) \tag{6.1}$$

where γ_2 and γ_3 represents spontaneous emission to other modes. The even and odd mode $c_e(x)$ is introduced is to simplify Eq. (2.1) which takes on the form,

$$c_e(x) = \frac{1}{\sqrt{2}}[c_R(x) + c_L(-x)] \tag{6.2}$$

$$c_o(x) = \frac{1}{\sqrt{2}}[c_R(x) - c_L(-x)] \tag{6.3}$$

Which after the transformation, H_p is separated into two modes H_e and H_o. It can shown that the odd mode Hamiltonian is non-interacting[27] we therefore only need to consider the even mode H_e

$$H_e = -ic \int \mathrm{d}x\, c_e^\dagger(x) \frac{\partial}{\partial x} c_e(x) \tag{6.4}$$

The interaction Hamiltonian is given by with $S_+ = |2\rangle\langle 1|$,

$$H_{int} = g \int \mathrm{d}x\, \delta(x)\, (S_- c_e^\dagger(x) + S_+ c_e(x))s \tag{6.5}$$

6.2 Solutions to the Hamiltonian

The scattering eigenstates of the system's Hamiltonian is given by

$$|E\rangle = \int \mathrm{d}x\, f_1(x) a_e^\dagger(x)|0,1\rangle + f_2|0,2\rangle + f_3|0,3\rangle \tag{6.6}$$

where

$$f_1(x) = \frac{1}{\sqrt{2\pi}}(\theta(-x) + t_k\theta(x))e^{ikx} \tag{6.7}$$

The coefficients can be found by equating $H|E\rangle = E|E\rangle$ where $E = ck$ is the energy of the incoming photon. We obtain the following system of equations

$$\begin{aligned} -ic\frac{\partial}{\partial x}f_1(x) + g\delta(x)f_2 &= ckf_1(x) \\ gf_1(0) + (E_2 - i\gamma_2/2)f_2 + \frac{\Omega}{2}f_3 &= ckf_2 \\ (E_2 - \Delta - i\gamma_3/2)f_3 + \frac{\Omega}{2}f_2 &= ckf_3 \end{aligned} \tag{6.8}$$

The transmission coefficient can be found by substituting the third equation of Eq. (6.8) into the second. After a simple algebraic manipulation we obtain,

$$t_k = \frac{[ck - (E_2 - \Delta - i\gamma_3)][ck - (E_2 - i\gamma_2) - i\Gamma/2] - \Omega^2/4}{[ck - (E_2 - \Delta - i\gamma_3)][ck - (E_2 - i\gamma_2) + i\Gamma/2] - \Omega^2/4} \tag{6.9}$$

Given the transmission coefficient, the excitation probabilities can be easily found

$$f_2 = \frac{1}{\sqrt{2\pi}}\frac{ic(t_k - 1}{\sqrt{c\Gamma}} \tag{6.10}$$

$$f_3 = \frac{1}{\sqrt{2\pi}}\frac{ic\Omega(t_k - 1)}{2(ck + \Delta)} \tag{6.11}$$

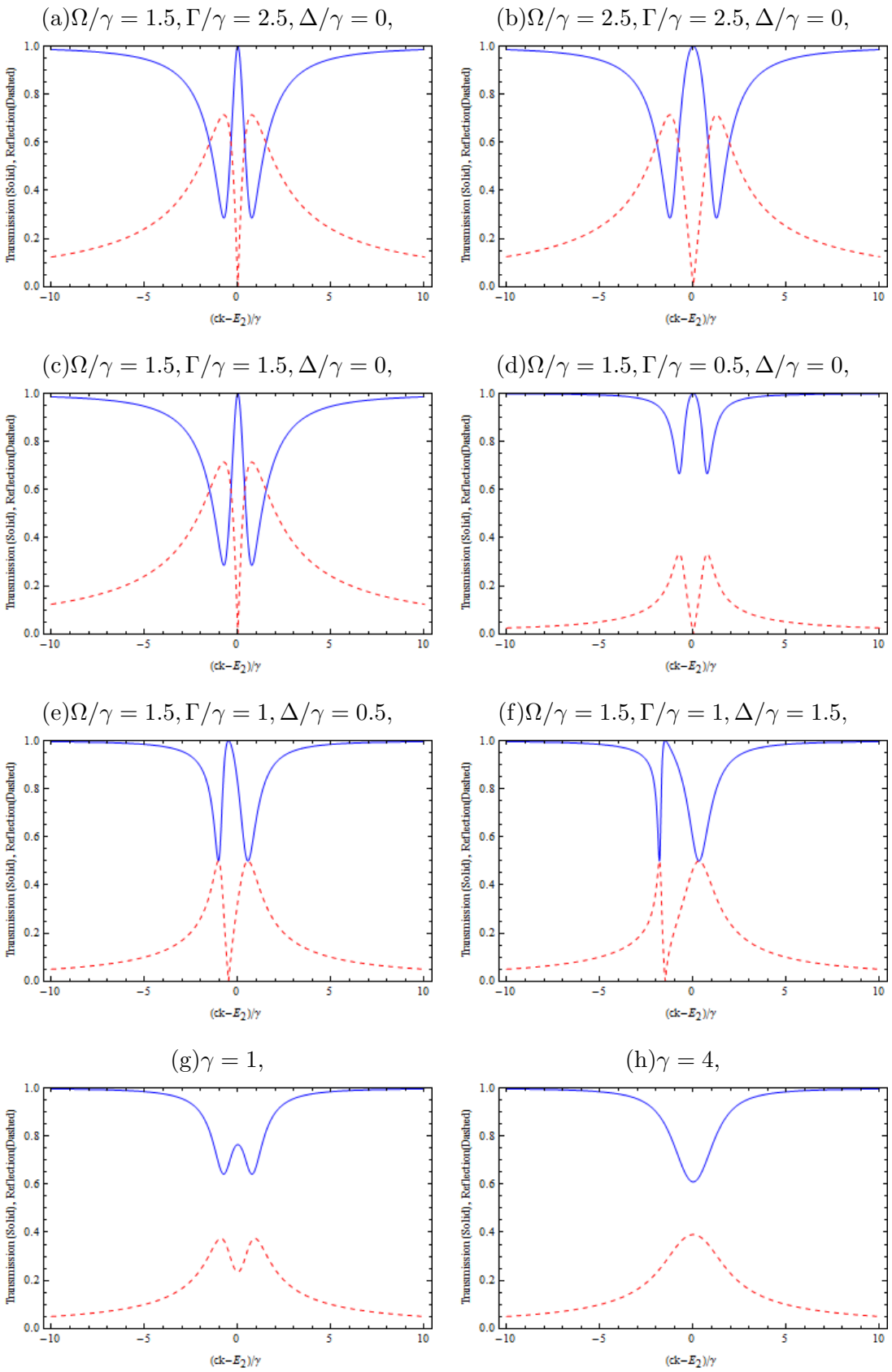

Figure 6.2: Transmission (solid line) and reflection (dashed line) spectra of the drive Λ system for different parameters. $\gamma = \gamma_3/\gamma_2$

From Fig. (6.2), we observe that the width of the transparency window is determined by the strength of the driving field, Ω. The two dips in the transmission spectra are controlled by the coupling strength between the atom and waveguide, Γ. The detuning parameter, Δ causes the spectra to be asymmetrical, and the spectra is symmetric only at $\Delta = 0$. The atom can only be fully transparent only when $|3\rangle$ is a metastable state, $\gamma_3 = 0$. When the γ increases, the transparency of the atom decreases until the EIT window vanishes.

Chapter 7

Two photons in a waveguide

In this section, we study study the scattering problem of two photons in a waveguide interacting with a two level system. Compared to the single photon scenario, it is harder to understand the mechanics behind photon-photon interactions in a waveguide. The standard solution was using the Lippmann-Schwinger formalism where the interaction Hamiltonian was assumed to be switched on adiabatically from the distant past($t \to -\infty$) and swtiched off after a long period of time($t \to \infty$). The output states and the interacting states at $t = 0$ were computed using the Lippmann-Schwinger equations.[19] Using the Bethe-ansatz method, the interacting state is first derived and used to form the scattering matrix. After which, the input and output states were derived from the interacting states.[26]

7.1 One mode Hamiltonian

Starting with the decoupled one-mode Hamilitonian of Eq. (6.4) and Eq. (6.5), we apply the same technique to a single photon eigenstate in the e-mode,

$$|E_k\rangle = \int \mathrm{d}x \left\{ \frac{1}{\sqrt{2\pi}} [\theta(-x) + t_k \theta(x)] e^{ikx} c_e^\dagger + e_k S_+ \right\} |0, g\rangle \tag{7.1}$$

And we obtain the following equations

$$-iv_g(t_k - 1) + Ve_k = 0$$
$$\frac{1}{2}(1 + t_k) = (E_k - E_e)e_k$$

From the equations of motion, we can obtain the transmission coefficient for an e-mode photon,

$$t_k = \frac{k - \Omega - i\Gamma/2}{k - \Omega + i\Gamma/2} \tag{7.2}$$

with $v_g = 1, E_k = k, \Omega = E_e - E_g, V^2 = \Gamma$. To convert the transmission coefficient to the left and right mode, We first define the S matrix which is the linear combination of S matrix the "e" subspace, S_e and the S matrix the "o" subspace,

$$S = \sum_k t_k |k\rangle_{ee}\langle k| + |k\rangle_{oo}\langle k| \tag{7.3}$$

Where $|k\rangle_{e/o} = \int dx\, \phi^k_{int}(x) c^\dagger_{e/o}(x)|0, g\rangle$ is the normalised in-state in the "e" or "o" subspace respectively. For an incoming photon from the right, it can be expressed in terms of $|k\rangle_{e/o}$,

$$|k\rangle_R = \frac{1}{\sqrt{2}}(|k\rangle_e + |k\rangle_o) \tag{7.4}$$

the out state is

$$\begin{aligned} S|k\rangle_R &= \frac{1}{\sqrt{2}}\left(\sum_{k'} t_{k'}|k'\rangle_{ee}\langle k'|k\rangle_e + |k'\rangle_{oo}\langle k'|k\rangle_o\right) \\ &= \frac{1}{\sqrt{2}}(t_k|k\rangle_e + |k\rangle_o) \\ &= \frac{1}{2}\left[(t_k + 1)|k\rangle_R + (t_k - 1)|-k\rangle_L\right] \end{aligned} \tag{7.5}$$

From Eq. (7.5) we can see that the the left and right mode transmission and the reflection coefficient correspond to the coefficients calculated from the out-state.

$$t = \frac{t_k + 1}{2} = \frac{k - \Omega}{k - \Omega + i\Gamma/2}$$
$$r = \frac{t_k - 1}{2} = \frac{-i\Gamma/2}{k - \Omega + i\Gamma/2} \tag{7.6}$$

which agrees with the previous calculations in Section 1.

7.2 Two-photon eigenstate

The two-photon eigenstate of H_e has the following form,

$$|i^+\rangle = \left(\int \mathrm{d}x_1 \mathrm{d}x_2 \, g(x_1, x_2) \frac{1}{\sqrt{2}} c_e^\dagger(x_1) c_e^\dagger(x_2) + \int \mathrm{d}x \, e(x) c_e^\dagger(x) S_+ \right) |0, g\rangle \tag{7.7}$$

where $e(x)$ is the probability amplitude distribution of one photon while the atom is in the excited state. From $H_e|i^+\rangle = E|i^+\rangle$, we obtain the equation of motions.

$$\left(-i\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2} - E \right) g(x_1, x_2) + \frac{V}{\sqrt{2}} [e(x_1)\delta(x_2) + e(x_2)\delta(x_1)] = 0 \tag{7.8a}$$
$$\left[-i\frac{\partial}{\partial x} - (E - \Omega) \right] e(x) + \frac{V}{\sqrt{2}} [g(0, x) + g(x, 0)] = 0 \tag{7.8b}$$

To solve for the equations of motion, we first look at the locations of the interactions, which occurs at $x_1, x_2 = 0$. If we plot the 2 photon wavefunction, the x and y axis will be the boundaries where the interaction occurs. Applying the equations of motion between the adjacent boundaries where $x_2 < x_1 < 0$ and $x_2 > x_1$ gives the boundary conditions. It can be achieved by integrating Eq. (7.8a) over x_2 over an infinitesimally small interval, $\{-\epsilon, \epsilon\}$,

$$\lim_{\epsilon \to 0} -i \int_{-\epsilon}^{\epsilon} dx_2 \, \frac{\partial}{\partial x_2} g(x_1, x_2) - i \int_{-\epsilon}^{\epsilon} dx_2 \, \frac{\partial}{\partial x_1} g(x_1, x_2) - E \int_{-\epsilon}^{\epsilon} dx_2 \, g(x_1, x_2) + \frac{V}{\sqrt{2}} \int_{-\epsilon}^{\epsilon} dx_2 \, [e(x_1)\delta(x_2) + e(x_2)\delta(x_1)] = 0$$
$$\lim_{\epsilon \to 0} -i \frac{\partial}{\partial x_2} g(x_1, \epsilon)(2\epsilon) - i[g(x_1, \epsilon) - g(x_1, -\epsilon)] - E g(x_1, \epsilon)(2\epsilon) + \frac{V}{\sqrt{2}} e(x_2) = 0$$

$$i[g(x_1,0^+)-g(x_1,0^-)]+\frac{V}{\sqrt{2}}e(x_1)=0$$
$$g(x_1,0^+)=g(x_1,0^-)-i\frac{V}{\sqrt{2}}e(x_1) \tag{7.9}$$

Since we can define $g(0,x)\equiv 1/2[g(0^+,x)+g(0^-,x)]=g(x,0)\equiv 1/2[g(x,0^+)+g(x,0^-)]$ and $x\equiv x_1-x_2$, we can rewrite Eq. (7.8b),

$$\left[-i\frac{\partial}{\partial x_1}-(E-\Omega)\right]e(x_1)+\frac{V}{\sqrt{2}}[g(x_1,0^+)+g(x_1,0^-)]=0 \tag{7.10}$$

By substituting Eq. 7.9 into Eq. (7.10), we obtain a first-order differential equation

$$\frac{\partial}{\partial x_1}e(x_1)-i(E-\Omega-i\Gamma/2)e(x_1)=-i\sqrt{2}Vg(x_1,0^-) \tag{7.11}$$

Solving Eq. (7.11) gives us the solution

$$e(x_1)=ce^{i((E-\Omega)+i\Gamma/2)x_1}+e^{i((E-\Omega)+i\Gamma/2)x_1}\int_{-\infty}^{x_1}dxe^{-i((E-\Omega)+i\Gamma/2)x}[-i\sqrt{2}Vg(x_1,0^-)] \tag{7.12}$$

When the coupling strength is zero, the probability of one of the photons exciting the atom is zero to, therefore we can say that $c=0$.

$$e(x_1)=e^{i((E-\Omega)+i\Gamma/2)x_1}\int_{-\infty}^{x_1}dxe^{-i(E-\Omega)+i\Gamma/2)x}[-i\sqrt{2}Vg(x,0^-)] \tag{7.13}$$

On the boundary of $x_2>x_1$ and $x_2>x_1>0$, the boundary conditions are,

$$g(0^+,x_2)=g(0^-,x_2)-i\frac{V}{\sqrt{2}}e(x_2) \tag{7.14}$$

$$\left[-i\frac{\partial}{\partial x_2}-(E-\Omega)\right]e(x_2)+\frac{V}{\sqrt{2}}[g(0^+,x_2)+g(0^-,x_2]=0 \tag{7.15}$$

Using Bethe ansatz, the two-photon wavefunction has the form,

$$g(x_1, x_2) = \begin{cases} B_3 e^{ikx_1+ipx_2} + A_3 e^{ikx_2+ipx_1} \\ \quad \text{In region } 3, x_1 < x_2 < 0 \\ B_2 e^{ikx_1+ipx_2} + A_2 e^{ikx_2+ipx_1} \\ \quad \text{In region } 2, x_1 < 0, x_2 > 0 \\ B_1 e^{ikx_1+ipx_2} + A_1 e^{ikx_2+ipx_1} \\ \quad \text{In region } 1, x_2 > x_1 > 0 \end{cases} \tag{7.16}$$

The wave function in other regions is defined by boson symmetry, $g(x_1, x_2) = g(x_2, x_1)$. The coefficients of the wavefunction can be found be applying the boundary conditions.

Starting with region 3 where $x_1 < x_2 < 0$, we have

$$g(x_1, 0^-) = B_3 e^{ikx_1} + A_3 e^{ikx_2} \tag{7.17}$$

which we can substitute into Eq. (7.13) to find the excitation amplitude distribution

$$e(x_1) = \sqrt{2}V \left(\frac{B_3 e^{ikx_1}}{p - \Omega + i\Gamma/2} + \frac{A_3 e^{ikx_1}}{k - \Omega + i\Gamma/2} \right) \tag{7.18}$$

and inserting it into Eq. (7.10) to obtain

$$g(x_1, 0^+) = t_p B_3 e^{ikx_1} + t_k A_3 e^{ipx_1} \tag{7.19}$$

Therefore, from region 3 to region 2, one can tell that the coefficients changes by a transmission factor.

$$B_2 = t_p B_3 \quad A_2 = t_k A_3 \tag{7.20}$$

Physically, it means that from region 3 to region 2, x_2 interact with the atom which give rise to a transmission amplitude. We can see that when x_2 has momentum p/k, it acquires a transmission coefficient $t_{p/k}$,

Applying the same technique for the wavefunction in region 2,

$$g(0^-, x_2) = t_p B_3 e^{ipx_2} + t_k A_3 e^{ikx_2} \tag{7.21}$$

And substituting to find the excitation amplitude distribution

$$e(x_2) = \sqrt{2}V \left(\frac{t_p B_3^{ipx_2}}{k - \Omega + i\Gamma/2} + \frac{t_k A_3 e^{ikx_2}}{p - \Omega + i\Gamma/2} \right) \tag{7.22}$$

and inserting it into Eq. (7.14) to obtain

$$g(0^+, x_2) = t_p t_k (B_3 e^{ipx_2} + A_3 e^{ikx_2}) \tag{7.23}$$

Therefore, from region 3 to region 1, one can tell that the coefficients changes by a transmission factor.

$$B_1 = t_k t_p B_3 \quad A_1 = t_p t_k A_3 \tag{7.24}$$

To find the constraints of A_3 and B_3, we need to introduce another boundary conditions which involves the excitation distribution

$$e(0^+) = e(0^-) \tag{7.25}$$

From Eq. (7.18) and Eq. (7.22) which are the excitation amplitudes from region 3 and region 2, we can obtain the ratio of A_3 and B_3

$$\frac{B_3}{A_3} = \frac{k - p - i\Gamma/2}{k - p + i\Gamma/2} \tag{7.26}$$

From the previous calculations of the interacting eigenstate $|i^+\rangle$, we can extract information about the in-state, $|i\rangle$ and the out-state,

$$|i\rangle = \int \mathrm{d}x_1 \mathrm{d}x_2 \, g_i(x_1, x_2) \frac{1}{\sqrt{2}} c_e^\dagger(x_1) c_e^\dagger(x_2) |0, g\rangle$$
$$|f_i\rangle = \int \mathrm{d}x_1 \mathrm{d}x_2 \, g_f(x_1, x_2) \frac{1}{\sqrt{2}} c_e^\dagger(x_1) c_e^\dagger(x_2) |0, g\rangle \tag{7.27}$$

7.3 Two-photon bound state

Intuitively, we can see that for an atom at $x = 0$, the two photon wavefunction for the in-state has to lie in the quadrant where $g_i(x_1 < 0, x_2 < 0)$ and the two photon wavefunction for the out-state has to lie in the quadrant where $g_f(x_1 > 0, x_2 > 0)$.

$$\begin{aligned} g_i(x_1, x_2) &= (A_3 e^{ikx_1 + ipx_2} + B_3 e^{ikx_2 + ipx_1})\theta(x_1 - x_2) + (B_3 e^{ikx_1 + ipx_2} + A_3 e^{ikx_2 + ipx_1})\theta(x_2 - x_1) \\ &\propto [(k - p + i\Gamma/2)e^{ikx_1 + ipx_2} + (k - p - i\Gamma/2)e^{ikx_2 + ipx_1}]\theta(x_1 - x_2) + \\ &[(k - p - i\Gamma/2)e^{ikx_1 + ipx_2} + (k - p + i\Gamma/2)e^{ikx_2 + ipx_1})]\theta(x_2 - x_1) \\ &= (k - p)(e^{ikx_1 + ipx_2} + e^{ikx_2 + ipx_1}) + i\Gamma(e^{ikx_1 + ipx_2} - e^{ikx_2 + ipx_1})\mathrm{sgn}(x_1 - x_2) \\ &\propto (k - p)S_{k,p}(x_1, x_2) + i\Gamma A_{k,p}(x_1, x_2) \end{aligned} \tag{7.28}$$

where $S_{k,p}(x_1, x_2)$ and $A_{k,p}(x_1, x_2)$ are

$$\begin{aligned} \langle x_1, x_2 | S_{k,p} \rangle = S_{k,p}(x_1, x_2) &= \frac{1}{2\pi} \frac{1}{\sqrt{2}} (e^{ikx_1 + ipx_2} + e^{ikx_2 + ipx_1}) \\ &= \frac{1}{2\pi} \frac{1}{\sqrt{2}} e^{i(k+p)} e^{(x_1 + x_2)/2} \left[e^{i(k-p)(x_1 - x_2)/2} + e^{-i(k-p)(x_1 - x_2)/2} \right] \\ &= \frac{\sqrt{2}}{2\pi} e^{iEx_c} \cos(\Delta x) \end{aligned} \tag{7.29}$$

and

$$\begin{aligned}\langle x_1, x_2|A_{k,p}\rangle = A_{k,p}(x_1,x_2) &= \frac{1}{2\pi}\frac{1}{\sqrt{2}}(e^{ikx_1+ipx_2}+e^{ikx_2+ipx_1})\mathrm{sgn}(x)\\ &= \frac{1}{2\pi}\frac{1}{\sqrt{2}}e^{i(k+p)}e^{(x_1+x_2)/2}\left[e^{i(k-p)(x_1-x_2)/2}-e^{-i(k-p)(x_1-x_2)/2}\right]\mathrm{sgn}(x)\\ &= \frac{\sqrt{2}i}{2\pi}e^{iEx_c}\sin(\Delta x)\mathrm{sgn}(x) \qquad (7.30)\end{aligned}$$

where $|S/A_{k,p}\rangle = \int dx\, S/A_{k,p}(x_1,x_2)c_e^\dagger(x_1)c_e^\dagger(x_2)|0,g\rangle$ respectively. It can be shown that although $|S_{k,p}\rangle$ and $|A_{k,p}\rangle$ are complete, they are not orthogonal to each other.

$$\langle S_{k_1,p_1}|A_{k_2,p_2}\rangle = \frac{i}{\pi}\delta(E_1-E_2)(2\Delta_2)\frac{\mathcal{P}}{\Delta_2^2-\Delta_1^2} \qquad (7.31)$$

where $\mathcal{P}$ denotes Cauchy principal value

Similarly, we find the wavefunction of the out-state to be,

$$g_f(x_1,x_2) \propto t_k t_p g_i(x_1,x_2) \qquad (7.32)$$

7.4 Scattering matrix in ee subspace

We can therefore proceed to build the two-photon S matrix, S_{ee}. We first define the normalised Wiegmann-Andrei state[3, 30]

$$|W_{k,p}\rangle_{ee} = \frac{1}{\sqrt{(k-p)^2+\Gamma^2}}(k-p)|S_{k,p}\rangle + i\Gamma|A_{k,p}\rangle = \frac{1}{\sqrt{4\Delta^2+\Gamma^2}}2\Delta|S_{k,p}\rangle + i\Gamma|A_{k,p}\rangle \qquad (7.33)$$

Since they are the eigenstate of S_{ee} matrix, with eigenvalues, $t_k t_p$ which can be easily inferred from Eq. (7.32), the in and out state are thus

$$|in\rangle = \int dx_1 dx_2\, W_{k,p}(x_1,x_2)\frac{1}{\sqrt{2}}c_e^\dagger(x_1)c_e^\dagger(x_2)|0,g\rangle \qquad (7.34)$$

$$|f_i\rangle = \int dx_1 dx_2\, t_k t_p W_{k,p}(x_1, x_2) \frac{1}{\sqrt{2}} c_e^\dagger(x_1) c_e^\dagger(x_2)|0, g\rangle \tag{7.35}$$

With

$$\begin{aligned} W_{k,p}(x_1, x_2) &= \langle x_1, x_2 | W_{k,p}\rangle_{ee} \\ &= \frac{\sqrt{2}}{2\pi} e^{iEx_c}[2\Delta\cos(\Delta x) - \Gamma \mathrm{sgn}(x)\sin(\Delta x)] \end{aligned} \tag{7.36}$$

Even though Wiegmann-Andrei state were derived from the Bethe-ansatz method, they are incomplete. Therefore, we need to find the remaining eigenstates to span the two-photon Hilbert space. To check whether $|W_{k,p}\rangle_{ee}$ span the two-photon Hilbert space, we can start with the state, $|S_{k,p}\rangle_{ee}$ and project out onto all the $|W_{k,p}\rangle_{ee}$ components

$$|\delta_{k,p}\rangle = |S_{k,p}\rangle_{ee} - \sum_{k' \leq p'} \langle W_{k',p'}|S_{k,p}\rangle_{ee} |W_{k',p'}\rangle_{ee} \tag{7.37}$$

if $|W_{k,p}\rangle_{ee}$ is complete, then the state should yield $|\delta_{k,p}\rangle_{ee} = 0$. The calculation shows that indeed the Wiegmann-Andrei state is indeed incomplete with $|\delta_{k,p}\rangle \propto |B_E\rangle_{ee}$

$$|B_E\rangle_{ee} \equiv \int dx_1 dx_2\, B_E(x_1, x_2) \frac{1}{\sqrt{2}} c_e^\dagger(x_1) c_e^\dagger(x_2)|0, g\rangle \tag{7.38}$$

with

$$\begin{aligned} B_E(x_1, x_2) &= \langle x_1, x_2 | B_E\rangle_{ee} \\ &= \sqrt{\frac{\Gamma}{4\pi}} e^{iEx_c - (\Gamma/2|x|)} \end{aligned} \tag{7.39}$$

From the previous derivation of the Wiegmann-Andrei state, we had assume that the photons momenta were distinct, $k < p$. Therefore the complete the two-photon Hilbert, we need to consider the regime where k p or a bound state where the photons are so close together the two level system

treats the incoming photons as a single photon. One feature of $B_E(x_1, x_2)$ is that when $x \to \pm\infty$, $B_E(x_1, x_2)| \to 0$, therefore $B_E(x_1, x_2)$ is a bound state.

Starting from the ansatz that the two-photon wavefunction in region $x_1 < x_2 < 0$ takes the form

$$g(x_1, x_2) = e^{iEx_c - (\Gamma/2x)} = e^{i(E-i\Gamma/2)x_1/2} e^{i(E+i\Gamma/2)x_2/2} \tag{7.40}$$

and using the same techniques previously, we can show that the two-photon bound state is an eigenstate of the S matrix, with eigenvalue

$$t_E = \frac{E - 2\Omega - 2i\Gamma}{E - 2\Omega + 2i\Gamma} \tag{7.41}$$

To construct the S_{ee} scattering matrix, we use the definition of the S-matrix $\sum_{|i\rangle} |f_i\rangle\langle i|$. From Eq. (7.35) and (7.34)

$$S_{ee} = \sum_{k \le p} t_k t_p |W_{k,p}\rangle\langle W_{k,p}| + \sum_E t_E |B_E\rangle\langle B_E| \tag{7.42}$$

With the scattering matrix defined, we can find the real-space representation of the out-state in the ee subspace.

$${}_{ee}\langle x_1, x_2|f\rangle_{ee} = e^{iEx_c} \frac{\sqrt{2}}{2\pi} \left(t_k t_p \cos(\Delta x) - \frac{4\Gamma^2}{4\Delta^2 - (E - 2\Omega + i\Gamma)^2} e^{i(E-2\Omega)|x|/2 - \Gamma|x|/2} \right) \tag{7.43}$$

After obtaining the solution to the scattering matrix in the ee subspace, we need to find the transport properties of the two photon in the left and right mode. Therefore we need to consider the following in-state

$$|i\rangle_{RR} = \int \mathrm{d}x_1 \mathrm{d}x_2 \, \frac{1}{2\pi} \frac{1}{\sqrt{2}} (e^{ikx_1 + ipx_2} + e^{ikx_2 + ipx_1}) \frac{1}{\sqrt{2}} c_R^\dagger(x_1) c_R^\dagger(x_2) |0, g\rangle \tag{7.44}$$

Using the transformation equations of Eq. (6.2) and Eq. (6.3), we can decompose the in-state into ee, oo and eo subspace, which we can compute the scattering states in these subspaces and finally transform them back to RR, LL and RL spaces.

$$\begin{aligned} |f\rangle &= S|i\rangle_{RR} \\ &= S_{ee}|i\rangle_{ee} + S_{oo}|i\rangle_{oo} + S_{eo}|i\rangle_{eo} \end{aligned} \tag{7.45}$$

After transforming back to RR, LL and RL spaces, we obtain

$$\begin{aligned} |f\rangle = &\int \mathrm{d}x_1\mathrm{d}x_2\, t_2(x_1,x_2)\frac{1}{\sqrt{2}}c_R^\dagger(x_1)c_R^\dagger(x_2)|0,g\rangle \\ &+ \int \mathrm{d}x_1\mathrm{d}x_2\, r_2(x_1,x_2)\frac{1}{\sqrt{2}}c_L^\dagger(x_1)c_L^\dagger(x_2)|0,g\rangle \\ &+ \int \mathrm{d}x_1\mathrm{d}x_2\, rt(x_1,x_2)c_R^\dagger(x_1)c_L^\dagger(x_2)|0,g\rangle \end{aligned} \tag{7.46}$$

where

$$t_2(x_1,x_2) = e^{iEx_c}\frac{\sqrt{2}}{2\pi}\left(\overline{t_k}\,\overline{t_p}\cos(\Delta x) - \frac{4\Gamma^2}{4\Delta^2-(E-2\Omega+i\Gamma)^2}e^{i(E-2\Omega)|x|/2-\Gamma|x|/2}\right) \tag{7.47}$$

$$r_2(x_1,x_2) = e^{-iEx_c}\frac{\sqrt{2}}{2\pi}\left(\overline{r_k}\,\overline{r_p}\cos(\Delta x) - \frac{4\Gamma^2}{4\Delta^2-(E-2\Omega+i\Gamma)^2}e^{i(E-2\Omega)|x|/2-\Gamma|x|/2}\right) \tag{7.48}$$

$$rt(x_1,x_2) = \frac{1}{2\pi}e^{(E/2)x}\left(\overline{t_k}\,\overline{r_p}e^{2i\Delta x_c} + \overline{r_k}\,\overline{t_p}e^{-2i\Delta x_c} - \frac{4\Gamma^2}{4\Delta^2-(E-2\Omega+i\Gamma)^2}e^{i(E-2\Omega)|x|/2-\Gamma|x|/2}\right) \tag{7.49}$$

where $\overline{t_k}\,\overline{t_p}$ and $\overline{r_k}\,\overline{r_p}$ are the single photon transmission coefficient in the left and right mode, Eq. (7.6). $x = x_1 - x_2$, $x_c = (x_1+x_2)/2$, $E = k+p$ and $\Delta = (k-p)/2$

We can plot $|t_2(x_1,x_2)|^2$, $|r_2(x_1,x_2)|^2$ and $|rt(x_1,x_2)|^2$ with x_1-x_2 and x_1+x_2 as the x-axis. By varying total energy detuning, $\delta E = E - 2\Omega$ and their energy difference, Δ.

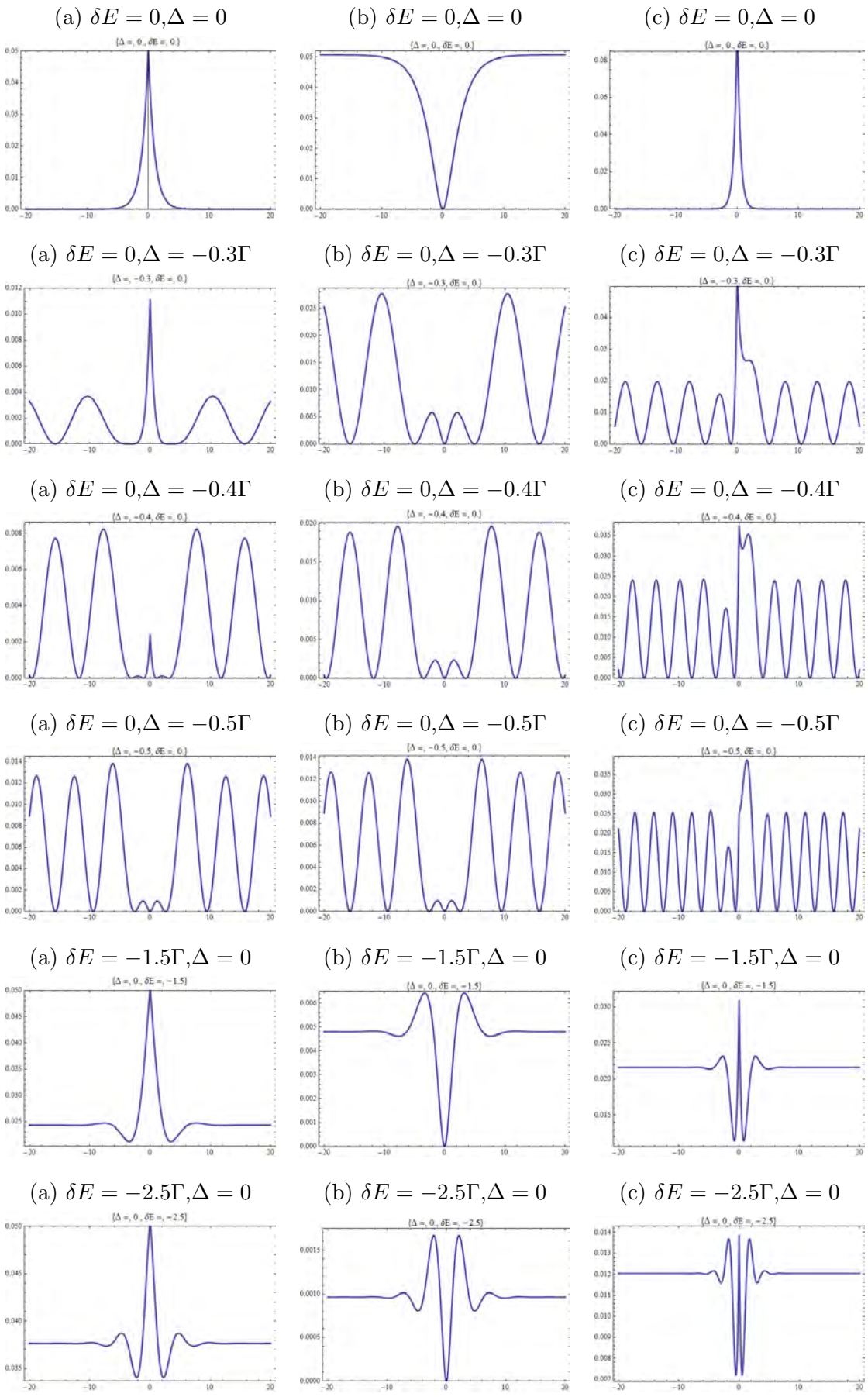

Figure 7.1: $|t_2(x_1, x_2)|^2$, $|r_2(x_1, x_2)|^2$ and $|rt(x_1, x_2)|^2$ with different detuning, δE and energy difference, δ

$|t_2(x_1, x_2)|^2$, $|r_2(x_1, x_2)|^2$ and $|rt(x_1, x_2)|^2$ are the probabilities amplitudes of the two photon in the forward direction, backward direction and two photons going in different directions respectively after interacting with the atom. From Fig. (7.1), the probability amplitude of $|t_2(x_1, x_2)|^2$ decreases to zero at $|x_1 - x_2| = 0$ as we set the 2 photons to be on resonance with the atom and increasing their energy differences. $|r_2(x_1, x_2)|^2$ remains zero at at $|x_1 - x_2| = 0$ when we tune the energy difference but probability of the two photons moving in the opposite directions is non-zero. Hence the photons change from bunching to anti-bunching and the atom is able to induce a repulsion in the two photon.

Chapter 8

Applications

Having solved the scattering problems, we will discuss the applications of atom photon interactions in a waveguide. In this chapter, we will explore the possibilities of implementing this class of scattering process in a atomic bragg mirror or even a scalable quantum gate.

8.1 Atomic Bragg Mirror

In a normal optical fiber, the refractive indices of the core and of the cladding do not change along the length of the fiber. Any deviation from this leads to scattering of the propagating light and to additional loss in the optical fiber. On the other hand, if we induce a periodic modulation of refractive index along the length in the core of the optical fiber, such a device, referred to as an Fiber Bragg Grating(FBG), exhibits very interesting spectral properties.[16]

When broadband light or a set of wavelengths are incident on an FBG, only the wavelength corresponding to the Bragg wavelength will get strongly reflected; the other wavelengths just get transmitted to the output. It can be shown by plotting the reflection amplitude.[16]

$$r = \frac{\sinh^2(L\sqrt{\kappa^2 - \sigma^2})}{\cosh^2(L\sqrt{\kappa^2 - \sigma^2}) - \left(\frac{\sigma}{\kappa}\right)^2} \tag{8.1}$$

where L is the periodicity of the grating and κ and σ are coupling coefficients.

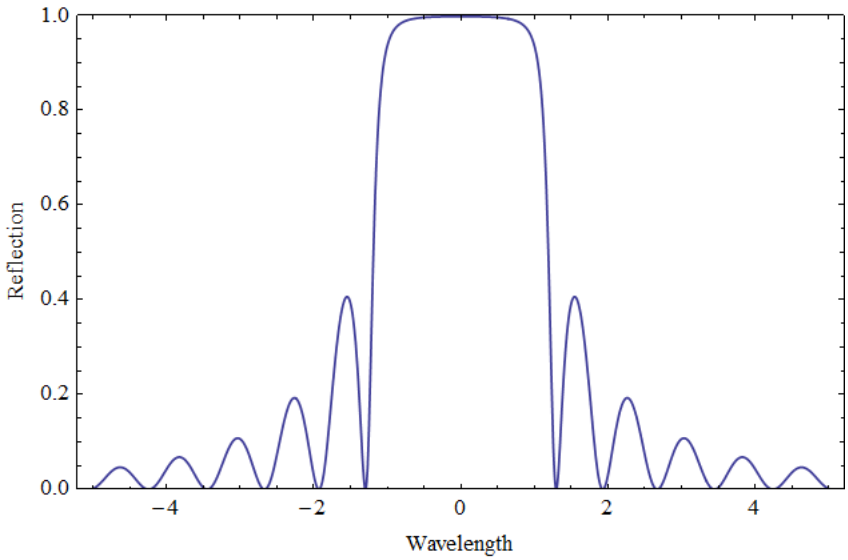

Figure 8.1: Reflection spectrum plot against wavelength where Bragg's wavelength, Λ_B is centered at the origin.

By comparing the spectral properties of Fibre Bragg Grating and N-atoms in waveguide, we can see that atoms form a 'Bragg mirror' and function in the same way as the Fibre Bragg Grating. It can be used for scalable quantum information network as photons are used to relay information over a network and the atom can be used to store information.[9]

8.2 Scalable quantum gate

A theoretical scheme for a two qubit entangling gate has been shown recently using the Lieb-Liniger model.[7] Let us rst consider the case of two spinless bosons in one dimension with a delta function interaction[18]

$$H = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + 2\Gamma\delta(x_1 - x_2) \tag{8.2}$$

where Γ is the coupling strength between the two bosons. The eigenfunctions of the model were computed by Lieb and Liniger. Using the techniques from the two photons case, we can extract the eigenvalues of the scattering matrix for the interaction for particles with momenta $p_1 < p_2$

$$S(p_1, p_2) = \frac{p_2 - p_1 - i\Gamma}{p_2 - p_1 + i\Gamma} \tag{8.3}$$

From the scattering process, we can show that it forms a quantum gate. We can call the photon moving towards the right as photon A and the photon moving towards the left as photon B. As we are considering indistinguishable particles labelled by momenta, all right moving photons are photon A and left moving are photon B even after the interactions. After applying the scattering matrix to the initial states,

$$\begin{aligned}
|0\rangle_A|0\rangle_B &\to \frac{p_0 + p_0 - i\Gamma}{p_0 + p_0 + i\Gamma}|0\rangle_A|0\rangle_B = -|0\rangle_A|0\rangle_B \\
|0\rangle_A|1\rangle_B &\to \frac{p_0 + p_1 - i\Gamma}{p_0 + p_1 + i\Gamma}|0\rangle_A|1\rangle_B = |0\rangle_A|1\rangle_B \\
|1\rangle_A|0\rangle_B &\to \frac{p_1 + p_0 - i\Gamma}{p_1 + p_0 + i\Gamma}|1\rangle_A|0\rangle_B = |1\rangle_A|0\rangle_B \\
|1\rangle_A|1\rangle_B &\to \frac{p_1 + p_1 - i\Gamma}{p_1 + p_1 + i\Gamma}|1\rangle_A|1\rangle_B = |1\rangle_A|1\rangle_B
\end{aligned} \tag{8.4}$$

where particles with momentum p_0 is labelled $|0\rangle_{A/B}$ and particles with momentum p_1 is labelled $|1\rangle_{A/B}$. We assume the $0 < p_0 \ll \Gamma$ and $p_1 \gg \Gamma$. From the resultant states of the particles we can see that the scattering produces a Controlled-NOT gate, which is a maximally-entangled gate which the two photons are initially in the state $\frac{1}{2}(|0\rangle_A + |1\rangle_A) \otimes (|0\rangle_B + |1\rangle_B)$.

From Eq. (7.42), we can spot the similarities between the scattering eigenvalues. However, it is a non-trivial issue to map the scattering matrix from the two photon case to the Lieb-Liniger model due to the bound state term.

Chapter 9

Conclusion

In summary we have introduced the introduced the mechanics of atom-photons in a waveguide in the interest of quantum information processing and many-body physics. We derived the transport properties of waveguide with a single photon, 2 level system. Fano-lineshape was produced using the transfer matrix method when additional mirrors surround the atom which agrees to Fano's theory where a asymmetrical lineshape will be produced with a discrete system is coupled to a continuum. A bandgap with energy proportional to the periodicities of the atom is formed when there is an infinite number of atoms in the waveguide. For a finite amount of atom in the waveguide, we found that the spectral properties of the system is similar to that of the Fibre Bragg Grating. However, when the atoms have random periodicities, we find that the photon is localised and is largely dependent on the coupling strength of the atoms to the waveguide.

In a driven three level Λ system, we derived the transport quantities using the same techniques but we have used a different Hamiltonian and eigenfunction. From Fig. (6.2), the properties of EIT atom still shows when it is embedded in a waveguide. We have derived the scattering matrix for two photons and found that there is a bound state when we checked for completeness, which gives rise to the precedence where we should always check for completeness for any kind of system in a waveguide formulated from Bethe ansatz.

It has been shown experimentally that it is possible to implement a system when the atom is coupled to the waveguide, the next breakthrough will be to find the map between the scattering

eigenvalues mentioned in the previous chapters. Another possibility can be to create a library similar to QuTip[15] in Python to where it will simplify the calculations for different kinds of systems. The basic mechanism of the interactions are well-understood and we need only to change the Hamiltonian of the atoms and the scattering eigenfunction.

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