Black Holes and Fluid Dynamics

by

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Abstract

This work re-examines Klemm and Maiorana's 2014 paper *Fluid dynamics on ultrastatic spacetimes and dual black holes* by replicating and extending their findings. In that paper, the black holes in 3+1 dimensions dual (via the Anti–de Sitter/Conformal Field Theory correspondence) to equilibrium fluids in ultrastatic manifolds in 2+1 dimensions with constant curvature in the spatial sections were constructed and classified. These black holes are all found within the Carter–Plebański class of solutions. After replicating their work, a deeper analysis of the Carter–Plebański class was conducted. This deeper analysis places a black hole dual that was merely mentioned in passing in context as well as discovering a black hole dual that was missing in the earlier classification, with both then explicitly constructed. Thus, this work fully classifies the black holes in 3+1 dimensions dual to equilibrium fluids in ultrastatic manifolds in 2+1 dimensions with constant curvature in the spatial sections.

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Contents

Chapter 1

Introduction

Anti–de Sitter/Conformal Field Theory (AdS/CFT) correspondence is a correspondence between two physical theories, a theory of gravity in the bulk of asymptotically Anti-de Sitter space in n dimensions and a conformal field theory of $n-1$ dimensions on the boundary of the bulk. It was first proposed by Juan Maldacena in his landmark paper [1], which has now over 10 000 citations. It is also known as holography, as it invokes the idea of a 2-D holographic plate storing a 3-D image.

While it was originally proposed in the context of string theory, it has been extended to beyond that, with applications in condensed matter, quantum information, particle physics and many more [2].

Anti-de Sitter space is an empty space with a negative cosmological constant, while conformal field theories are (quantum) field theories that are conformally invariant, which means that the physics of the theory remains unchanged under conformal transformations, which are transformations that preserves angles locally.

This remarkable duality between gravity and gauge theories has been utilised in many fields, as general relativity can be applied to calculate results in strongly coupled field theories, where perturbative methods fail.

In this thesis, we will deal in gravity/fluid duality, which is a subset of AdS/CFT. This is essentially embedding black holes in AdS space, with a (conformal) fluid living on the boundary of it. There has been much work on this [3, 4], with the Navier–Stokes equations being derived up to second order in [5] for arbitrary dimensions.

Fluid dynamics is the natural behaviour of most, if not all field theories in the long wavelength limit, and it is thus unsurprising that we can obtain fluid dynamics as a dual to gravity. There is a hope that this duality would bring a new perspective to both fields, including issues such as turbulence, which is still not understood fully despite its importance. Despite the constrains on the duality, it is believed that insights gained from solving these solutions can be applicable to general relativity and fluid dynamics as a whole.

In [6], Klemm and Maiorana claimed that they have fully classified the equilibrium flows on ultrastatic manifolds, in 2+1 dimensions, of constant curvature in the spatial sections as well as their black hole duals in 3+1 dimensions. The equilibrium flows were classified by unique flows (up to an isometry) on the spatial sections, via the aid of a proof that equates Killing fields on the spatial sections to equilibrium flows on the manifold. Only one type of flow was found on the 2-sphere, while two were found for the Euclidean plane and three for the 2-hyperbola.

The black hole duals to these flows were then constructed from the Carter–Plebański class of solutions, which is a very general solution of black holes, and includes, among others, the Kerr–Newman black hole solution within it. This was done using an analysis via a quartic $P(p)$ within the Carter–Plebański metric. In explicitly constructing these dual black holes, a few new solutions to the Carter–Plebański class was found. One finding of this work is that the conformal boundaries of the dual black holes matches that of the physical region in which the fluid velocity does not exceed the speed of light.

For many of the constructed solutions, and in AdS/CFT in general, the black holes dealt with are extended, in contrast with the familiar compact black holes. For example, in contrast with the standard Kerr black hole, there is also a hyperbolic Kerr black hole, with the event horizon being hyperbolic, unlike the spherical case.

We re-examine Klemm and Maiorana's work [6] in this thesis, and find that they are missing a dual black hole. This is filled by considering a quantity known as the NUT charge, which has been ignored in the original classification. The NUT charge is a quantity that exists in certain solutions to Einstein's equations of general relativity [7, 8]. It is generally considered to be an undesired quantity, as it leads to closed timelike curves, at least in the traditional black hole solutions, and is thus usually considered unphysical.

The NUT charge has been shown by Griffiths and Podolský [9] to not be the parameter n in the Carter–Plebański class of black holes, but rather that of a new parameter l, which arises from a shift and rescaling of the coordinate p. This can be seen in the Kerr–Newman–NUT–(anti-)de Sitter metric in [9]. The confusion over the nature of the NUT charge arose because when the rotation a and cosmological constant Ω is set to 0, $n = l$, which was historically how NUT charge was studied [7, 8].

With this understanding, a black hole dual to a case that is missing in the current classification in [6] was formulated. This completes the “dictionary” relating this specific type of flows to their black holes, allowing for further study.

The organisation of the paper is as follows: firstly, a few results in conformal fluid dynamics and fluids in ultrastatic manifolds are proved. Next, these results are used in classifying the possible fluid flows on ultrastatic manifolds with constant curvature on the spatial sections. The black holes dual to the fluid flows were then found, with the help of a classification of the quartic $P(p)$ in the Carter–Plebański metric.

A new form of $P(p)$, found in [9], is then used to find the remaining two cases that were not found in the earlier classification of $P(p)$. The first case is a well known result, and mentioned but not detailed in [6], but the second is new.

Chapter 2

Conformal Fluid Dynamics in Ultrastatic Spacetimes

2.1 Conformal Fluid Dynamics

We now introduce conformal fluid dynamics, which is our CFT half of AdS/CFT. In this thesis, we will use the indices i, j, k for spatial and μ, ν, ρ for arbitrary dimensions.

A conformal transformation is given by a Weyl rescaling, where the metric is rescaled [10]

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{2.1}$$

with Ω being a smooth, strictly positive function. It follows that

$$\tilde{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu}. \tag{2.2}$$

The Christoffel symbols transforms with an additional term C,

$$\tilde{\Gamma} = \Gamma + C, \tag{2.3}$$

with

$$C^{\rho}{}_{\mu\nu} = \frac{1}{2}\tilde{g}^{\rho\beta}(\nabla_\mu \tilde{g}_{\nu\beta} + \nabla_\nu \tilde{g}_{\mu\beta} - \nabla_\beta \tilde{g}_{\mu\nu}). \tag{2.4}$$

The objective is to rewrite C in terms of the original metric, so from

$$\begin{aligned}\nabla_\rho \tilde{g}_{\mu\nu} &= \nabla_\rho(\Omega^2 g_{\mu\nu}) \\ &= 2\Omega g_{\mu\nu}\nabla_\rho\Omega + \Omega^2\nabla_\rho g_{\mu\nu} \\ &= 2\Omega g_{\mu\nu}\nabla_\rho\Omega, \end{aligned} \tag{2.5}$$

utilizing the fact that $\nabla_\rho g_{\mu\nu} = 0$. Thus,

$$\begin{aligned} C^\rho{}_{\mu\nu} &= \frac{1}{2}\tilde{g}^{\rho\beta}(\nabla_\mu\tilde{g}_{\nu\beta} + \nabla_\nu\tilde{g}_{\mu\beta} - \nabla_\beta\tilde{g}_{\mu\nu}) \\ &= \frac{1}{2}\Omega^{-2}g^{\rho\beta}(2\Omega g_{\nu\beta}\nabla_\mu\Omega + 2\Omega g_{\mu\beta}\nabla_\nu\Omega - 2\Omega g_{\mu\nu}\nabla_\beta\Omega) \\ &= \Omega^{-1}(\delta^\rho_\nu\nabla_\mu\Omega + \delta^\rho_\mu\nabla_\nu\Omega - g^{\rho\beta}g_{\mu\nu}\nabla_\beta\Omega). \end{aligned} \tag{2.6}$$

2.1.1 Conformal Transformation Rule for the Energy-Momentum Tensor

Fluids obey the conservation laws for the energy-momentum tensor $T^{\mu\nu}$ and the charge current J^μ,

$$\nabla_\mu T^{\mu\nu} = 0, \qquad \nabla_\mu J^\mu = 0. \tag{2.7}$$

In this thesis, we are considering fluids in equilibrium. For these fluids, there can be no dissipative effects, because the entropy of such a system is constant in time. Thus, the fluid will have the form of a perfect fluid, and will be shearless and expansionless [4, 6].

The energy-momentum tensor must transform with a conformal weight w [10], if the theory is conformal, as follows:

$$\tilde{T}^{\mu\nu} = \Omega^w T^{\mu\nu}. \tag{2.8}$$

Note that w is not a tensor index.

Hence,

$$
\begin{aligned}
\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} &= \nabla_\mu \tilde{T}^{\mu\nu} + C^\mu{}_{\mu\rho}\tilde{T}^{\rho\nu} + C^\nu{}_{\mu\rho}\tilde{T}^{\mu\rho} \\
&= \nabla_\mu(\Omega^w T^{\mu\nu}) + C^\mu{}_{\mu\rho}(\Omega^w T^{\rho\nu}) + C^\nu{}_{\mu\rho}(\Omega^w T^{\mu\rho}) \\
&= \Omega^w \nabla_\mu T^{\mu\nu} + w\Omega^{w-1}T^{\mu\nu}\nabla_\mu\Omega + \Omega^{-1}(\delta^\mu_\rho \nabla_\mu\Omega + \delta^\mu_\mu \nabla_\rho\Omega - g^{\mu q}g_{\mu\rho}\nabla_q\Omega)\Omega^w T^{\rho\nu} \\
&\quad + \Omega^{-1}(\delta^\nu_\rho \nabla_\mu\Omega + \delta^\nu_\mu \nabla_\rho\Omega - g^{\nu q}g_{\mu\rho}\nabla_q\Omega)\Omega^w T^{\mu\rho} \\
&= \Omega^w \nabla_\mu T^{\mu\nu} + \Omega^{w-1}[wT^{\mu\nu}\nabla_\mu\Omega + (\nabla_\rho\Omega + d\nabla_\rho\Omega - \delta^q_\rho\nabla_q\Omega)T^{\rho\nu} \\
&\quad + (\delta^\nu_\rho \nabla_\mu\Omega + \delta^\nu_\mu \nabla_\rho\Omega - g^{\nu q}g_{\mu\rho}\nabla_q\Omega)T^{\mu\rho}] \\
&= \Omega^w \nabla_\mu T^{\mu\nu} + \Omega^{w-1}[wT^{\mu\nu}\nabla_\mu\Omega + T^{\mu\nu}\nabla_\mu\Omega + dT^{\mu\nu}\nabla_\mu\Omega - T^{\mu\nu}\nabla_\mu\Omega \\
&\quad + T^{\mu\nu}\nabla_\mu\Omega + T^{\nu\mu}\nabla_\mu\Omega - T^\mu{}_\mu\nabla^\nu\Omega] \\
&= \Omega^w \nabla_\mu T^{\mu\nu} + \Omega^{w-1}[(w+d+2)T^{\mu\nu}\partial_\mu\Omega - T^\mu{}_\mu\partial^\nu\Omega],
\end{aligned}
\tag{2.9}
$$

where the symmetry of $T^{\mu\nu}$, $\nabla_\mu\Omega = \partial_\mu\Omega$ and the shorthand notation of $g^{\mu\nu}\nabla_\mu = \nabla^\nu$ are used to produce the final result. Note that d is the dimensions of the spacetime manifold considered.

A conformal fluid would still have to obey the conservations laws in (2.7), thus we conclude that

$$w = -d - 2 \tag{2.10}$$

and

$$T^\mu{}_\mu = 0. \tag{2.11}$$

From the definition of the energy momentum stress tensor of a perfect fluid,

$$T^{\mu\nu} = (\rho + \mathcal{P})u^\mu u^\nu + \mathcal{P}g^{\mu\nu}, \tag{2.12}$$

with ρ being the density and $\mathcal{P}$ being the pressure, we get

$$
\begin{aligned}
T^\mu{}_\mu &= (\rho + \mathcal{P})u^\mu u_\mu + \mathcal{P}\delta^\mu_\mu \\
&= -(\rho + \mathcal{P}) + \mathcal{P}d.
\end{aligned}
\tag{2.13}
$$

From (2.11) and (2.13), we get the equation of state

$$\rho = (d-1)\mathcal{P}, \tag{2.14}$$

yielding the stress tensor of a conformal perfect fluid,

$$T^{\mu\nu} = \mathcal{P}(d\, u^\mu u^\nu + g^{\mu\nu}). \tag{2.15}$$

2.1.2 Conformal Transformation Rule for the Velocity Vector

Now, we show how a fluid with a velocity profile u^μ transforms conformally. We know that for all timelike objects,

$$u^\mu u_\mu = g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \frac{ds^2}{d\tau^2} = -\frac{d\tau^2}{d\tau^2} = -1. \tag{2.16}$$

Thus, given that

$$\begin{aligned} \tilde{u}^\mu \tilde{u}_\mu = -1 &= \tilde{u}^\mu \tilde{g}_{\mu\nu} \tilde{u}^\nu \\ &= \tilde{u}^\mu \Omega^2 g_{\mu\nu} \tilde{u}^\nu \\ &= u^\mu g_{\mu\nu} u^\nu, \end{aligned} \tag{2.17}$$

we conclude that

$$\tilde{u}^\mu = \Omega^{-1} u^\mu. \tag{2.18}$$

2.2 Equlibrum fluids in Ultrastatic spacetimes

An ultrastatic spacetime can always be represented in a coordinate system such that

$$ds^2 = -dt^2 + \bar{g}_{ij} dx^i dy^j, \tag{2.19}$$

where we introduce the notation $\bar{g}$ to refer to indices running across spatial dimensions only. This can be thought of as curved space with flat time.

A fluid flowing in such a manifold would have a velocity

$$u^\mu = \gamma(1, v^i), \tag{2.20}$$

which leads to

$$u^\mu u_\mu = \gamma^2(-1 + \bar{g}_{ij} v^i v^j). \tag{2.21}$$

Because of (2.16), it can be concluded that

$$\gamma^2 = \frac{1}{1-\bar{g}_{ij}v^i v^j} = \frac{1}{1-v^2}, \tag{2.22}$$

with $v^2 = \bar{g}_{ij}v^i v^j$.

2.2.1 Relating the fluid flows to Killing fields

We will now prove a rather lengthy result that will link equilibrium fluid flows in ultrastatic spacetimes to that of Killing fields in the spatial sections of the manifolds. This will allow us to classify the fluid flows in the next chapter.

As we are considering equilibrium flows,

$$\partial_t u^\mu = 0, \tag{2.23}$$

i.e. it does not change with time.

We now consider the term

$$\nabla_\mu u^\nu = \partial_\mu u^\nu + \Gamma^\nu_{\mu\rho}u^\rho. \tag{2.24}$$

Breaking it up into its constituent components, we have

$$\begin{aligned}\nabla_t u^\mu &= \partial_t u^\mu + \Gamma^\mu_{t\nu}u^\nu \\ &= \Gamma^\mu_{t\nu}u^\nu \\ &= \frac{1}{2}g^{\rho\mu}(\partial_t g_{\nu\rho} + \partial_\nu g_{\rho t} - \partial_\rho g_{t\nu})u^\nu \\ &= 0,\end{aligned} \tag{2.25}$$

due to $g_{\mu\nu}$ being independent of time, $g_{it} = 0$ and $g_{tt} = -1$, because of the ultrastatic spacetime.

It also implies that

$$\begin{aligned}\nabla_\mu u^t &= \partial_\mu u^t + \Gamma^t_{\mu\nu}u^\nu \\ &= \partial_\mu u^t + \frac{1}{2}g^{pt}(\partial_\mu g_{\nu p} + \partial_\nu g_{p\mu} - \partial_p g_{\mu\nu})u^\nu \\ &= \partial_\mu u^t,\end{aligned} \tag{2.26}$$

following the same reasoning as (2.25). Lastly,

$$
\begin{aligned}
\nabla_i u^j &= \partial_i u^j + \Gamma^j_{i\mu} u^\mu \\
&= \partial_i u^j + \Gamma^j_{ik} u^k + \Gamma^j_{it} u^t \\
&= \partial_i u^j + \Gamma^j_{ik} u^k + \frac{1}{2} g^{\nu j}(\partial_i g_{t\nu} + \partial_t g_{\nu i} - \partial_\nu g_{it}) u^t \\
&= \partial_i u^j + \Gamma^j_{ik} u^k \\
&= \partial_i u^j + \frac{1}{2} g^{\mu j}(\partial_i g_{k\mu} + \partial_k g_{\mu i} - \partial_\mu g_{ik}) u^k \\
&= \partial_i u^j + \frac{1}{2}[\bar{g}^{lj}(\partial_i \bar{g}_{kl} + \partial_k \bar{g}_{li} - \partial_l \bar{g}_{ik}) + g^{tj}(\partial_i g_{kt} + \partial_k g_{ti} - \partial_t g_{ik})] u^k \\
&= \partial_i u^j + \frac{1}{2} \bar{g}^{lj}(\partial_i \bar{g}_{kl} + \partial_k \bar{g}_{li} - \partial_l \bar{g}_{ik}) u^k \\
&= \partial_i u^j + \bar{\Gamma}^j_{ik} u^k \\
&= \bar{\nabla}_i u^j = \bar{\nabla}_i(\gamma v^j) = v^j \partial_i \gamma + \gamma \bar{\nabla}_i v^j,
\end{aligned}
\tag{2.27}
$$

again using the same reasoning as in (2.25), and using the notation $\bar{\nabla}$ and $\bar{\Gamma}$ in the way first introduced in (2.19).

Next, another useful result will be shown.

$$
\begin{aligned}
\partial_i(\gamma^2) &= \partial_i(1 - v^2)^{-1} \\
2\gamma \partial_i \gamma &= (1 - v^2)^{-2} \partial_i v^2 \\
&= \gamma^4 \partial_i v^2 \\
\partial_i \gamma &= \frac{1}{2} \gamma^3 \partial_i v^2 \\
&= \frac{1}{2} \gamma^3 \partial_i (\bar{g}_{jk} v^j v^k) \\
&= \frac{1}{2} \gamma^3 (\bar{g}_{jk} v^j \partial_i v^k + \bar{g}_{jk} v^k \partial_i v^j + v^j v^k \partial_i \bar{g}_{jk}) \\
&= \frac{1}{2} \gamma^3 (2 v_j \partial_i v^j + v^j v^k \partial_i \bar{g}_{jk}).
\end{aligned}
\tag{2.28}
$$

Consider the term

$$
\begin{aligned}
2 v_j \bar{\nabla}_i v^j &= 2 v_j \partial_i v^j + 2 v_j \bar{\Gamma}^j_{ik} v^k \\
&= 2 v_j \partial_i v^j + v_j \bar{g}^{lj}(\partial_i \bar{g}_{kl} + \partial_k \bar{g}_{li} - \partial_l \bar{g}_{ik}) v^k \\
&= 2 v_j \partial_i v^j + v^l v^k (\partial_i \bar{g}_{kl} + \partial_k \bar{g}_{li} - \partial_l \bar{g}_{ik}) \\
&= 2 v_j \partial_i v^j + v^l v^k \partial_i \bar{g}_{kl}.
\end{aligned}
\tag{2.29}
$$

Therefore,

$$\partial_i \gamma = \gamma^3 v_j \bar{\nabla}_i v^j. \tag{2.30}$$

Consider a shearless and expansionless fluid. From the expansion equation [11],

$$\theta = \nabla_\mu u^\mu, \tag{2.31}$$

using it in an ultrastatic spacetime,

$$\begin{aligned}\theta &= \nabla_\mu u^\mu \\ &= \nabla_t u^t + \nabla_i u^i \\ &= \nabla_i u^i \\ &= v^i \partial_i \gamma + \gamma \bar{\nabla}_i v^i,\end{aligned} \tag{2.32}$$

which arises from the usage of (2.25) and (2.27).

With the aid of the previous results, we will now calculate the shear tensor, given by [11]

$$\sigma^{\mu\nu} = \frac{1}{2}(P^{\mu\rho}\nabla_\rho u^\nu + P^{\nu\rho}\nabla_\rho u^\mu) - \frac{1}{d-1}\theta P^{\mu\nu}, \tag{2.33}$$

where

$$P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu. \tag{2.34}$$

As earlier, the tensor will be broken into its components, with

$$\begin{aligned}\sigma^{tt} &= \frac{1}{2}(P^{t\rho}\nabla_\rho u^t + P^{t\rho}\nabla_\rho u^t) - \frac{1}{d-1}\theta P^{tt} \\ &= (g^{t\rho} + u^t u^\rho)\partial_\rho u^t - \frac{1}{d-1}(v^i\partial_i\gamma + \gamma\bar{\nabla}_i v^i)(g^{tt} + u^t u^t) \\ &= (g^{tt} + u^t u^t)\partial_t u^t + (g^{ti} + u^t u^i)\partial_i u^t - \frac{1}{d-1}(v^i\partial_i\gamma + \gamma\bar{\nabla}_i v^i)(-1+\gamma^2) \\ &= u^t u^i \partial_i u^t + \frac{1-\gamma^2}{d-1}(v^i\partial_i\gamma + \gamma\bar{\nabla}_i v^i) \\ &= \gamma^2 v^i \partial_i \gamma + \frac{\gamma^2}{d-1}(\frac{1}{\gamma^2} - 1)v^i\partial_i\gamma + \frac{\gamma^3}{d-1}(\frac{1}{\gamma^2} - 1)\bar{\nabla}_i v^i \\ &= \frac{\gamma^2}{d-1}(1 - v^2 - 1 + d - 1)v^i\partial_i\gamma + \frac{\gamma^3}{d-1}(1 - v^2 - 1)\bar{\nabla}_i v^i \\ &= \frac{\gamma^2}{d-1}(d - 1 - v^2)v^i\partial_i\gamma - \frac{v^2\gamma^3}{d-1}\bar{\nabla}_i v^i,\end{aligned} \tag{2.35}$$

$$\begin{aligned}
\sigma^{ti} &= \frac{1}{2}(P^{t\rho}\nabla_\rho u^i + P^{i\rho}\nabla_\rho u^t) - \frac{1}{d-1}\theta P^{ti} \\
&= \frac{1}{2}[(g^{t\rho} + u^t u^\rho)\nabla_\rho u^i + (g^{i\rho} + u^i u^\rho)\nabla_\rho u^t] - \frac{1}{d-1}(v^j\partial_j\gamma + \gamma\bar{\nabla}_j v^j)(g^{ti} + u^t u^i) \\
&= \frac{1}{2}[u^t u^j \nabla_j u^i + (\bar{g}^{ij} + u^i u^j)\partial_j u^t] - \frac{1}{d-1}(v^j\partial_j\gamma + \gamma\bar{\nabla}_j v^j)(u^t u^i) \\
&= \frac{1}{2}[\gamma^2 v^j(v^i\partial_j\gamma + \gamma\bar{\nabla}_j v^i) + (\bar{g}^{ij} + \gamma^2 v^i v^j)\partial_j\gamma] - \frac{1}{d-1}(v^j\partial_j\gamma + \gamma\bar{\nabla}_j v^j)\gamma^2 v^i \\
&= \frac{d-2}{d-1}\gamma^2 v^j v^i \partial_j\gamma + \frac{1}{2}\bar{g}^{ij}\partial_j\gamma + \gamma^3(\frac{1}{2}v^j\bar{\nabla}_j v^i - \frac{1}{d-1}v^i\bar{\nabla}_j v^j),
\end{aligned} \tag{2.36}$$

$$\begin{aligned}
\sigma^{ij} &= \frac{1}{2}(P^{i\rho}\nabla_\rho u^j + P^{j\rho}\nabla_\rho u^i) - \frac{1}{d-1}\theta P^{ij} \\
&= \frac{1}{2}[(g^{i\rho} + u^i u^\rho)\nabla_\rho u^j + (g^{j\rho} + u^j u^\rho)\nabla_\rho u^i] - \frac{1}{d-1}(v^k\partial_k\gamma + \gamma\bar{\nabla}_k v^k)(\bar{g}^{ij} + u^i u^j) \\
&= \frac{1}{2}[(\bar{g}^{ik} + \gamma^2 v^i v^k)\nabla_k u^j + (\bar{g}^{jk} + \gamma^2 v^j v^k)\nabla_k u^i] - \frac{1}{d-1}(v^k\partial_k\gamma + \gamma\bar{\nabla}_k v^k)(\bar{g}^{ij} + \gamma^2 v^i v^j) \\
&= \frac{1}{2}[(\bar{g}^{ik} + \gamma^2 v^i v^k)(v^j\partial_k\gamma + \gamma\bar{\nabla}_k v^j) + (\bar{g}^{jk} + \gamma^2 v^j v^k)(v^i\partial_k\gamma + \gamma\bar{\nabla}_k v^i)] \\
&\quad - \frac{1}{d-1}(v^k\partial_k\gamma + \gamma\bar{\nabla}_k v^k)(\bar{g}^{ij} + \gamma^2 v^i v^j) \\
&= (\frac{1}{2}\bar{g}^{ik}v^j + \frac{1}{2}\bar{g}^{jk}v^i - \frac{1}{d-1}\bar{g}^{ij}v^k)\partial_k\gamma + \gamma(\frac{1}{2}\bar{\nabla}^i v^j + \frac{1}{2}\bar{\nabla}^j v^i - \frac{1}{d-1}\bar{g}^{ij}\bar{\nabla}^k v^k) \\
&\quad + \frac{d-2}{d-1}\gamma^2 v^i v^j v^k\partial_k\gamma + \gamma^3(\frac{1}{2}v^i v^k\bar{\nabla}_k v^j + \frac{1}{2}v^j v^k\bar{\nabla}_k v^i - \frac{1}{d-1}v^i v^j\bar{\nabla}_k v^k).
\end{aligned} \tag{2.37}$$

Now note that

$$v^j\sigma^{ti} = \frac{d-2}{d-1}\gamma^2 v^i v^j v^k\partial_k\gamma + \frac{1}{2}v^j\bar{g}^{ik}\partial_k\gamma + \gamma^3(\frac{1}{2}v^j v^k\bar{\nabla}_k v^i - \frac{1}{d-1}v^i v^j\bar{\nabla}v^k). \tag{2.38}$$

We can now rewrite (2.37) in terms of (2.36), with the use of (2.38) with the following result:

$$\begin{aligned}
\sigma^{ij} &= v^i\sigma^{tj} + v^j\sigma^{ti} - \frac{d-2}{d-1}\gamma^2 v^i v^j v^k\partial_k\gamma + \frac{1}{d-1}\gamma^3 v^i v^j\bar{\nabla}_k v^k \\
&\quad - \frac{1}{d-1}\bar{g}^{ij}(v^k\partial_k\gamma + \gamma\bar{\nabla}_k v^k) + \frac{\gamma}{2}(\bar{\nabla}^i v^j + \bar{\nabla}^j v^i).
\end{aligned} \tag{2.39}$$

Given that the fluid is shearless, $\sigma^{\mu\nu} = 0$. Applying this to (2.35) results in

$$v^i\partial_i\gamma = \frac{v^2\gamma}{d-1-v^2}\bar{\nabla}_i v^i. \tag{2.40}$$

Now utilising $\theta = 0$ (expansionless) with (2.32) and (2.40),

$$\begin{aligned}\theta = 0 &= v^i \partial_i \gamma + \gamma \bar{\nabla}_i v^i \\ &= \frac{v^2\gamma}{d-1-v^2} \bar{\nabla}_i v^i + \gamma \bar{\nabla}_i v^i \\ &= \gamma\left(\frac{d-1}{d-1-v^2}\right) \bar{\nabla}_i v^i. \end{aligned} \tag{2.41}$$

As $\gamma(\frac{d-1}{d-1-v^2}) \neq 0$, this implies that

$$\bar{\nabla}_i v^i = 0. \tag{2.42}$$

Using the shearless property $\sigma^{\mu\nu} = 0$ on (2.39), and applying (2.40), followed by (2.42) gives

$$\begin{aligned}0 = \sigma^{ij} &= -\frac{d-2}{d-1}\gamma^2 v^i v^j \frac{v^2\gamma}{d-1-v^2}\bar{\nabla}_k v^k + \frac{1}{d-1}\gamma^3 v^i v^j \bar{\nabla}_k v^k \\ &\quad - \frac{1}{d-1}\bar{g}^{ij}\left(\frac{v^2\gamma}{d-1-v^2}\bar{\nabla}_k v^k + \gamma\bar{\nabla}_k v^k\right) + \frac{\gamma}{2}(\bar{\nabla}^i v^j + \bar{\nabla}^j v^i) \\ &= \frac{\gamma}{2}(\bar{\nabla}^i v^j + \bar{\nabla}^j v^i), \end{aligned} \tag{2.43}$$

from which we conclude that

$$\bar{\nabla}^i v^j + \bar{\nabla}^j v^i = 0, \tag{2.44}$$

which is the Killing equation on the spatial sections.

We will now verify our results by working in reverse. If we start from the Killing equation (2.44), we can extract (2.42) via

$$\bar{g}_{ij}(\bar{\nabla}^i v^j + \bar{\nabla}^j v^i) = 2\bar{\nabla}_i v^i = 0, \tag{2.45}$$

while if we consider (2.30), mutiplying it by v^i gives

$$\begin{aligned} v^i \partial_i \gamma &= \gamma^3 v_j v^i \bar{\nabla}_i v^j \\ &= \frac{\gamma^3}{2}(v_j v^i \bar{\nabla}_i v^j + v_i v^j \bar{\nabla}_j v^i) \\ &= \frac{\gamma^3}{2}(v_i v_j(\bar{\nabla}^i v^j + \bar{\nabla}^j v^i)) \\ &= 0. \end{aligned} \tag{2.46}$$

This along with (2.42), applied on (2.32) allows us to conclude that

$$\theta = v^i\partial_i\gamma + \gamma\bar{\nabla}_i v^i = 0, \tag{2.47}$$

which tells us that a Killing flow is expansionless.

(2.42) and (2.46) then tells us by (2.35) that

$$\sigma^{tt} = \frac{\gamma^2}{d-1}(d-1-v^2)v^i\partial_i\gamma - \frac{v^2\gamma^3}{d-1}\bar{\nabla}_i v^i = 0. \tag{2.48}$$

Applying (2.46) and (2.42), then (2.30) to (2.36) results in

$$\begin{aligned}
\sigma^{ti} &= \frac{d-2}{d-1}\gamma^2 v^j v^i\partial_j\gamma + \frac{1}{2}\bar{g}^{ij}\partial_j\gamma + \gamma^3(\frac{1}{2}v^j\bar{\nabla}_j v^i - \frac{1}{d-1}v^i\bar{\nabla}_j v^j) \\
&= \frac{1}{2}\bar{g}^{ij}\partial_j\gamma + \gamma^3\frac{1}{2}v^j\bar{\nabla}_j v^i \\
&= \gamma^3\frac{1}{2}\bar{g}^{ij}v_k\bar{\nabla}_j v^k + \gamma^3\frac{1}{2}v^j\bar{\nabla}_j v^i \\
&= \frac{\gamma^3}{2}(v_k\bar{\nabla}^i v^k + v_j\bar{\nabla}^j v^i) \\
&= \frac{\gamma^3}{2}v_j(\bar{\nabla}^i v^j + \bar{\nabla}^j v^i) = 0.
\end{aligned} \tag{2.49}$$

Finally, applying (2.49), (2.46) and (2.42) to (2.39) gives

$$\sigma^{ij} = \frac{\gamma}{2}(\bar{\nabla}^i v^j + \bar{\nabla}^j v^i) = 0. \tag{2.50}$$

(2.48), (2.49) and (2.50) tell us that a Killing flow is shearless.

We have now showed that a Killing flow on the spatial sections on the ultrastatic manifold means it is expansionless and shearless, as well as vice versa. Note that this result is independent of the fluid being conformal. As mentioned earlier, an equilibrium fluid would be expansionless and shearless, due to entropy being constant in time.

This is a significant result, as it allows us to reduce the problem of studying and classifying the various equilibrium fluid flows to that of the study of the Killing fields on the spatial sections, due to this equivalence. This will be used to great effect in the next chapter.

2.2.2 Relation of Pressure to γ

Now, we will prove another result that would be used in the later chapters. For a conformal fluid in equilibrium, the stress tensor has the form of (2.15). The pressure is constant with time, i.e. $\partial_t \mathcal{P} = 0$, and thus

$$\begin{aligned}\nabla_\mu T^{\mu\nu} &= \nabla_\mu(\mathcal{P}(du^\mu u^\nu + g^{\mu\nu})) \\ &= \partial_\mu \mathcal{P}(du^\mu u^\nu + g^{\mu\nu}) + d\mathcal{P}(u^\nu \nabla_\mu u^\mu + u^\mu \nabla_\mu u^\nu) \\ &= \partial_i \mathcal{P}(du^i u^\nu + g^{i\nu}) + d\mathcal{P} u^\mu \nabla_\mu u^\nu, \end{aligned} \tag{2.51}$$

where we note that

$$\nabla_\mu u^\mu = \nabla_t u^t + \nabla_i u^i = 0 \tag{2.52}$$

by (2.25), (2.27), (2.42) and (2.46).

Breaking up the term $u^\mu \nabla_\mu u^\nu$, we get

$$\begin{aligned} u^\mu \nabla_\mu u^t &= u^i \bar{\nabla}_i u^t = u^i \partial_i u^t \\ &= u^i \partial_i \gamma = \gamma v^i \partial_i \gamma = 0 \end{aligned} \tag{2.53}$$

and

$$\begin{aligned} u^\mu \nabla_\mu u^j &= u^i \bar{\nabla}_i u^j = \gamma v^i (v^j \partial_i \gamma + \gamma \bar{\nabla}_i v^j) \\ &= \gamma v^j v^i \partial_i \gamma + \gamma^2 v^i \bar{\nabla}_i v^j = \gamma^2 v^i \bar{\nabla}_i v^j, \end{aligned} \tag{2.54}$$

with the use of (2.25), (2.26), (2.27) and (2.46).

Applying these on (2.51), it becomes

$$\begin{aligned} \nabla_\mu T^{\mu t} &= \partial_i \mathcal{P}(du^i u^t + g^{it}) + d\mathcal{P} u^\mu \nabla_\mu u^t \\ &= du^i u^t \partial_i \mathcal{P} \\ &= d\gamma^2 v^i \partial_i \mathcal{P} \end{aligned} \tag{2.55}$$

and

$$\begin{aligned} \nabla_\mu T^{\mu j} &= \partial_i \mathcal{P}(du^i u^j + \bar{g}^{ij}) + d\mathcal{P} u^\mu \nabla_\mu u^j \\ &= \partial_i \mathcal{P}(du^i u^j + \bar{g}^{ij}) + d\mathcal{P} \gamma^2 v^i \bar{\nabla}_i v^j \\ &= d\gamma^2 v^i v^j \partial_i \mathcal{P} + \partial^j \mathcal{P} + d\mathcal{P} \gamma^2 v^i \bar{\nabla}_i v^j. \end{aligned} \tag{2.56}$$

Since the conservation equation, $\nabla_\mu T^{\mu\nu} = 0$, is obeyed, we conclude from (2.55) that

$$v^i \partial_i \mathcal{P} = 0, \tag{2.57}$$

and thus from (2.56)

$$\partial^j \mathcal{P} = -d\mathcal{P}\gamma^2 v^i \bar{\nabla}_i v^j. \tag{2.58}$$

Noting that from the Killing equation (2.44) we have

$$\bar{\nabla}^i v^j = -\bar{\nabla}^j v^i, \tag{2.59}$$

we then rewrite (2.58) with the aid of (2.30) into

$$\begin{aligned}
\partial_j \mathcal{P} &= -d\mathcal{P}\gamma^2 v^i \bar{\nabla}_i v_j \\
\frac{\partial_j \mathcal{P}}{\mathcal{P}} &= d\gamma^2 v^i \bar{\nabla}_j v_i \\
\partial_j \ln \mathcal{P} &= d\gamma^2 v^i \bar{\nabla}_j v_i \\
&= \frac{d\partial_j \gamma}{\gamma} \\
&= d\partial_j \ln \gamma.
\end{aligned} \tag{2.60}$$

This implies that

$$\mathcal{P} = \mathcal{P}_0 \gamma^d, \tag{2.61}$$

for some constant $\mathcal{P}_0$.

Chapter 3

Classifying the Different Equilibrium Fluid Flows

In this chapter, we classify the different equilibrium flows on ultrastatic spacetimes. We learnt in chapter 2 that Killing fields in the spatial sections are equivalent to equilibrium flows. If a flow can be mapped to another via an isometric transformation, which preserves the metric, the two flows are physically equivalent.

We denote the (continuous) isometry group of the spatial sections $(\Sigma, \bar{g})$ as $I(\Sigma)$, which is a Lie group [12] with a corresponding Lie algebra denoted as $i(\Sigma)$. The Lie algebra of this group is given by the vector space of the Killing fields, with the Lie bracket of two fields X and Y given by their commutator (as vector fields), i.e. $[X, Y] = XY - YX$. We note here that the adjoint representation of field Y under field X is also given by

$$Ad_X Y = [X, Y]. \tag{3.1}$$

The group is related to the algebra via the exponential map

$$X \quad \longmapsto \quad e^{tX}, \tag{3.2}$$

which maps X to the one-parameter subgroup parametrised by t it generates.

An isometry Ψ (in $I(\Sigma)$) can thus be written as

$$\Psi = e^{t^i X_i}, \tag{3.3}$$

where $\{X_i\}$ is a basis of independent Killing fields and t^i are parameters.

Given that the flows we consider are Killing, mapping a flow X to Y via an isometry Ψ can be written as [6]

$$\Psi e^{tX} \Psi^{-1} = e^{tY}, \tag{3.4}$$

or

$$\Psi X \Psi^{-1} = Y, \tag{3.5}$$

via differentiation with respect to t.

This means that the two flows X and Y are physically equivalent. Thus, to classify all the different flows, we merely need to find the minimum set of flows that can generate the full set of Killing flows via isometric transformations.

In this thesis, we will only consider ultrastatic manifolds with constant curvature in the spatial sections in 2+1 dimensions, i.e. the 2-sphere, the 2-hyperbola and the Euclidean plane. The unique flows will be found for each case via reducing the full set Killing flows into the minimum set using isometric transformations.

3.1 Euclidean Plane

We first start with the Euclidean plane. The metric is simply

$$ds^2 = -dt^2 + dx^2 + dy^2, \tag{3.6}$$

with all Killing fields in the spatial section made up of linear combinations of the following three ones[1]:

$$R = -y\partial_x + x\partial_y, \quad T_1 = \partial_y, \quad T_2 = \partial_x, \tag{3.7}$$

representing rotation and translations. Calculating commutation relations, we get

$$\begin{aligned}
[R, T_1] &= (-y\partial_x + x\partial_y)\partial_y - \partial_y(-y\partial_x + x\partial_y) \\
&= -y\partial_x\partial_y + x{\partial_y}^2 + y\partial_y\partial_x + \partial_x\partial_y y - x{\partial_y}^2 - {\partial_y}^2 x \\
&= \partial_x = T_2,
\end{aligned} \tag{3.8}$$

[1][6] has an inconsistency in the labelling of the independent Killing fields with respect to their commutator relations. Their final result is accurate though.

$$\begin{aligned}[R, T_2] &= (-y\partial_x + x\partial_y)\partial_x - \partial_x(-y\partial_x + x\partial_y) \\ &= -y{\partial_x}^2 + x\partial_y\partial_x + y{\partial_x}^2 + {\partial_x}^2 y - x\partial_x\partial_y - \partial_y\partial_x x \\ &= -\partial_y = -T_1\end{aligned} \tag{3.9}$$

and

$$[T_1, T_2] = \partial_y\partial_x - \partial_x\partial_y = 0, \tag{3.10}$$

remembering that partial derivatives commute.

Motivated by (3.5), we would like to work out the term

$$e^{a(m^1T_1+m^2T_2)}Re^{-a(m^1T_1+m^2T_2)}, \tag{3.11}$$

with a a constant, and $\sqrt{(m^1)^2 + (m^2)^2} = 1$.

Using the Campbell–Baker–Hausdorff formula, we have

$$e^X Y e^{-X} = e^{Ad_X}Y = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + ..., \tag{3.12}$$

and working out the commutator relations between some terms, we get

$$[m^1T_1 + m^2T_2, R] = -m^1T_2 + m^2T_1, \tag{3.13}$$

$$[m^1T_1 + m^2T_2, [m^1T_1 + m^2T_2, R]] = [m^1T_1 + m^2T_2, -m^1T_2 + m^2T_1] = 0, \tag{3.14}$$

allowing us to conclude that

$$e^{a(m^1T_1+m^2T_2)}Re^{-a(m^1T_1+m^2T_2)} = R + a(-m^1T_2 + m^2T_1). \tag{3.15}$$

Given the choice of

$$a = \frac{\beta}{\omega}, \quad m^1 = -\frac{\beta^2}{\beta}, \quad m^2 = \frac{\beta^1}{\beta}, \quad \beta = \sqrt{(\beta^1)^2 + (\beta^2)^2}, \tag{3.16}$$

we have

$$R + a(-m^1T_2 + m^2T_1) = R + \frac{\beta}{\omega}(\frac{\beta^2}{\beta}T_2 + \frac{\beta^1}{\beta}T_1), \tag{3.17}$$

which implies that for $\omega \neq 0$

$$e^{Ad_{a(m^1T_1+m^2T_2)}}\omega R = \omega R + \beta^1 T_1 + \beta^2 T_2, \tag{3.18}$$

i.e. as long as there is some rotation, all flows are equivalent to a purely rotating one about the origin.

3.1.1 Pure Rotation

Such a fluid would have a velocity of

$$u = \gamma(\partial_t + \omega\partial_\phi) \tag{3.19}$$

if we write in polar coordinates, with

$$\gamma = \frac{1}{\sqrt{1-\omega^2 r^2}}. \tag{3.20}$$

The fluid will only exist at

$$r < \frac{1}{|\omega|}, \tag{3.21}$$

due to speed of light considerations.

The energy momentum tensor would then be

$$\begin{aligned} T^{\mu\nu} &= \mathcal{P}(3u^\mu u^\nu + g^{\mu\nu}) \\ &= \mathcal{P}\left(3\begin{bmatrix} \gamma^2 & 0 & \gamma^2\omega \\ 0 & 0 & 0 \\ \gamma^2\omega & 0 & \gamma^2\omega^2 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{r^2} \end{bmatrix}\right) \\ &= \mathcal{P}\begin{bmatrix} 3\gamma^2-1 & 0 & 3\gamma^2\omega \\ 0 & 1 & 0 \\ 3\gamma^2\omega & 0 & \frac{3\gamma^2-2}{r^2} \end{bmatrix}, \end{aligned} \tag{3.22}$$

given that

$$\begin{aligned}3\gamma^2\omega^2 + \frac{1}{r^2} &= \frac{3\gamma^2\omega^2 r^2 + 1}{r^2} = \frac{\gamma^2(3\omega^2 r^2 + 1 - \omega^2 r^2)}{r^2} \\ &= \frac{\gamma^2(2\omega^2 r^2 + 1)}{r^2} = \frac{\gamma^2(3 - 2(1 - \omega^2 r^2))}{r^2} \\ &= \frac{3\gamma^2 - 2}{r^2}.\end{aligned} \tag{3.23}$$

3.1.2 Pure Translation

A purely translating fluid would have the velocity profile of

$$u = \gamma(\partial_t + \beta\partial_x), \tag{3.24}$$

with

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \tag{3.25}$$

where a change of coordinates via a rotation can bring this direction to any other purely translating flow. The fluid would live on the entire manifold, and the only limitation is that

$$\beta^2 < 1, \tag{3.26}$$

due to the speed of light.

The stress tensor is thus

$$\begin{aligned}T^{\mu\nu} &= \mathcal{P}(3u^\mu u^\nu + g^{\mu\nu}) \\ &= \mathcal{P}\left(3\begin{bmatrix}\gamma^2 & \gamma^2\beta & 0 \\ \gamma^2\beta & \gamma^2\beta^2 & 0 \\ 0 & 0 & 0\end{bmatrix} + \begin{bmatrix}-1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix}\right) \\ &= \mathcal{P}\begin{bmatrix}3\gamma^2 - 1 & 3\gamma^2\beta & 0 \\ 3\gamma^2\beta & 3\gamma^2\beta^2 + 1 & 0 \\ 0 & 0 & 1\end{bmatrix}.\end{aligned} \tag{3.27}$$

3.2 2-sphere

The metric is simply given by

$$ds^2 = -dt^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{3.28}$$

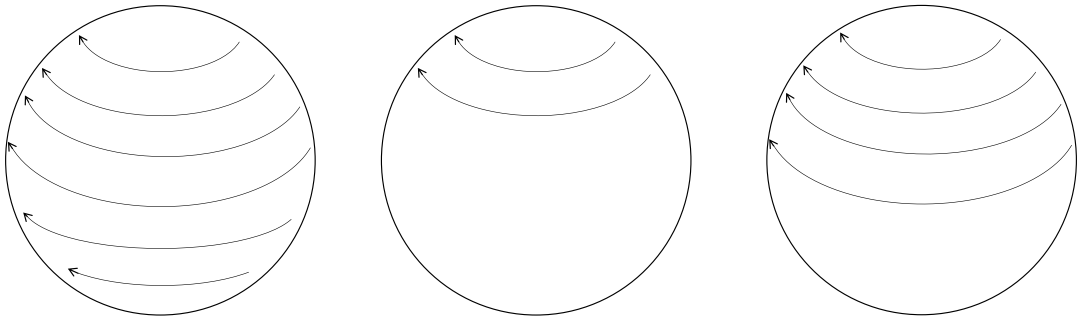

FIGURE 3.1: From left: Visualisation of fluid flowing over the whole 2-sphere, only over the polar region and only over the hemisphere. The region where the fluid flows is limited in the latter two cases due to the speed of light limit. Each of these cases have a black hole dual to it.

where a rotation can bring a Killing field to any other one, thus leaving only one distinct case.

3.2.1 Pure Rotation

For simplicity, we take that to be ∂_ϕ, giving us a fluid velocity of

$$u = \gamma(\partial_t + \omega\partial_\phi), \tag{3.29}$$

with

$$\gamma = \frac{1}{\sqrt{1 - \omega^2 r^2 \sin^2\theta}}. \tag{3.30}$$

The stress tensor then yields

$$\begin{aligned} T^{\mu\nu} &= \mathcal{P}(3u^\mu u^\nu + g^{\mu\nu}) \\ &= \mathcal{P}\left(3\begin{bmatrix} \gamma^2 & 0 & \gamma^2\omega \\ 0 & 0 & 0 \\ \gamma^2\omega & 0 & \gamma^2\omega^2 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix}\right) \\ &= \mathcal{P}\begin{bmatrix} 3\gamma^2 - 1 & 0 & 3\gamma^2\omega \\ 0 & \frac{1}{r^2} & 0 \\ 3\gamma^2\omega & 0 & \frac{3\gamma^2-2}{r^2\sin^2\theta} \end{bmatrix}. \end{aligned} \tag{3.31}$$

Speed of light considerations mean that

$$\omega^2 r^2 \sin^2\theta < 1, \tag{3.32}$$

so the fluid will cover the whole sphere when $|\omega| < \frac{1}{r}$, or extend from the polar regions from either pole when $|\omega| > \frac{1}{r}$ to the line $\sin\theta < \frac{1}{|\omega| r}$. The special case of $|\omega| = \frac{1}{r}$ would lead to the fluid covering half the sphere from either pole.

3.3 2-hyperbola

The metric of the 2-hyperbola is given by

$$ds^2 = -dt^2 + r^2(d\theta^2 + \sinh^2\theta d\phi^2). \tag{3.33}$$

The Poincaré disk coordinates will be used to analyse the fluid flows, so a short introduction will be given here. The Poincaré disk coordinates are

$$z = e^{i\phi}\tanh\frac{\theta}{2}, \quad \bar{z} = e^{-i\phi}\tanh\frac{\theta}{2}, \tag{3.34}$$

noting that to transform coordinates we have

$$g_{\acute{\mu}\acute{\nu}} = \frac{\partial x^{\mu}}{\partial \acute{x}^{\acute{\mu}}}\frac{\partial x^{\nu}}{\partial \acute{x}^{\acute{\nu}}} g_{\mu\nu}. \tag{3.35}$$

Working out the relevant information, we have

$$\theta = 2\tanh^{-1}\sqrt{z\bar{z}}, \quad \phi = \frac{i}{2}\ln\frac{\bar{z}}{z}, \quad \sinh^2\theta = \frac{4z\bar{z}}{(1-z\bar{z})^2}, \tag{3.36}$$

as well as

$$\frac{\partial\theta}{\partial z} = \frac{\bar{z}}{(1-z\bar{z})\sqrt{z\bar{z}}}, \quad \frac{\partial\theta}{\partial\bar{z}} = \frac{z}{(1-z\bar{z})\sqrt{z\bar{z}}}, \quad \frac{\partial\phi}{\partial z} = -\frac{i}{2z}, \quad \frac{\partial\phi}{\partial\bar{z}} = \frac{i}{2\bar{z}}. \tag{3.37}$$

Changing coordinates, we get

$$\begin{aligned}
\bar{g}_{ij} &= r^2 \begin{bmatrix} \frac{\partial\theta}{\partial z} & \frac{\partial\phi}{\partial z} \\ \frac{\partial\theta}{\partial\bar{z}} & \frac{\partial\phi}{\partial\bar{z}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sinh^2\theta \end{bmatrix} \begin{bmatrix} \frac{\partial\theta}{\partial z} & \frac{\partial\theta}{\partial\bar{z}} \\ \frac{\partial\phi}{\partial z} & \frac{\partial\phi}{\partial\bar{z}} \end{bmatrix} \\
&= r^2 \begin{bmatrix} (\frac{\partial\theta}{\partial z})^2 + (\frac{\partial\phi}{\partial z})^2\sinh^2\theta & \frac{\partial\theta}{\partial\bar{z}}\frac{\partial\theta}{\partial z} + \frac{\partial\phi}{\partial\bar{z}}\frac{\partial\phi}{\partial z}\sinh^2\theta \\ \frac{\partial\theta}{\partial\bar{z}}\frac{\partial\theta}{\partial z} + \frac{\partial\phi}{\partial\bar{z}}\frac{\partial\phi}{\partial z}\sinh^2\theta & (\frac{\partial\theta}{\partial\bar{z}})^2 + (\frac{\partial\phi}{\partial\bar{z}})^2\sinh^2\theta \end{bmatrix} \\
&= \frac{2r^2}{(1-z\bar{z})^2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\end{aligned} \tag{3.38}$$

with the metric of the ultrastatic manifold now

$$ds^2 = -dt^2 + \frac{4r^2}{(1-z\bar{z})^2} dz d\bar{z}, \tag{3.39}$$

with all Killing fields in the spatial section made up of linear combinations of the following three[2]:

$$\begin{aligned}
R &= i(z\partial_z - \bar{z}\partial_{\bar{z}}), \\
B_1 &= \frac{i}{2}(1+z^2)\partial_z - \frac{i}{2}(1+\bar{z}^2)\partial_{\bar{z}}, \\
B_2 &= \frac{1}{2}(1-z^2)\partial_z + \frac{1}{2}(1-\bar{z}^2)\partial_{\bar{z}}.
\end{aligned} \tag{3.40}$$

These have the commutation relations:

$$\begin{aligned}
[R, B_1] &= i\left(z\partial_z - \bar{z}\partial_{\bar{z}}\right)\left(\frac{i}{2}(1+z^2)\partial_z - \frac{i}{2}(1+\bar{z}^2)\partial_{\bar{z}}\right) \\
&\quad - \left(\frac{i}{2}(1+z^2)\partial_z - \frac{i}{2}(1+\bar{z}^2)\partial_{\bar{z}}\right) i(z\partial_z - \bar{z}\partial_{\bar{z}}) \\
&= -\frac{1}{2}\left(2z^2\partial_z + 2\bar{z}^2\partial_{\bar{z}} - (1+z^2)\partial_z - (1+\bar{z}^2)\partial_{\bar{z}}\right) \\
&= \frac{1}{2}(1-z^2)\partial_z + \frac{1}{2}(1-\bar{z}^2)\partial_{\bar{z}} = B_2,
\end{aligned} \tag{3.41}$$

[2] [6] again has an inconsistency in the labelling of the independent Killing fields with respect to their commutator relations. Their argument is thus somewhat modified here.

$$\begin{aligned}[R, B_2] &= i(z\partial_z - \bar{z}\partial_{\bar{z}})\left(\frac{1}{2}(1-z^2)\partial_z + \frac{1}{2}(1-\bar{z}^2)\partial_{\bar{z}}\right) \\ &\quad - \left(\frac{1}{2}(1-z^2)\partial_z + \frac{1}{2}(1-\bar{z}^2)\partial_{\bar{z}}\right) i(z\partial_z - \bar{z}\partial_{\bar{z}}) \\ &= \frac{i}{2}(-2z^2\partial_z + 2\bar{z}^2\partial_{\bar{z}} - (1-z^2)\partial_z + (1-\bar{z}^2)\partial_{\bar{z}}) \\ &= -\frac{i}{2}(1+z^2)\partial_z + \frac{i}{2}(1+\bar{z}^2)\partial_{\bar{z}} = -B_1, \end{aligned} \tag{3.42}$$

$$\begin{aligned}[B_1, B_2] &= \left(\frac{i}{2}(1+z^2)\partial_z - \frac{i}{2}(1+\bar{z}^2)\partial_{\bar{z}}\right)\left(\frac{1}{2}(1-z^2)\partial_z + \frac{1}{2}(1-\bar{z}^2)\partial_{\bar{z}}\right) \\ &\quad - \left(\frac{1}{2}(1-z^2)\partial_z + \frac{1}{2}(1-\bar{z}^2)\partial_{\bar{z}}\right)\left(\frac{i}{2}(1+z^2)\partial_z - \frac{i}{2}(1+\bar{z}^2)\partial_{\bar{z}}\right) \\ &= \frac{i}{4}\left((1+z^2)(-2z)\partial_z - (1+\bar{z}^2)(-2\bar{z})\partial_{\bar{z}} - (1-z^2)2z\partial_z + (1-\bar{z}^2)2\bar{z}\partial_{\bar{z}}\right) \\ &= \frac{i}{4}(-4z\partial_z + 4\bar{z}\partial_{\bar{z}}) = -R, \end{aligned} \tag{3.43}$$

where we note again that partial derivatives commute and are thus left out of the calculation to reduce clutter.

Now, consider the following term:

$$\begin{aligned} e^{\alpha R}(\omega R + \beta B_2)e^{-\alpha R} &= \omega R + \beta B_2 - \alpha\beta B_1 - \frac{1}{2!}\alpha^2\beta B_2 + \frac{1}{3!}\alpha^3\beta B_1 + \frac{1}{4!}\alpha^4\beta B_2 \\ &\quad - \frac{1}{5!}\alpha^5\beta B_1 - \frac{1}{6!}\alpha^6\beta B_2 + \frac{1}{7!}\alpha^7\beta B_1 + \frac{1}{8!}\alpha^8\beta B_2 - ... \\ &= \omega R + \beta B_2 - \frac{1}{2!}\alpha^2\beta B_2 + \frac{1}{4!}\alpha^4\beta B_2 - \frac{1}{6!}\alpha^6\beta B_2 + \frac{1}{8!}\alpha^8\beta B_2 - ... \\ &\quad - \alpha\beta B_1 + \frac{1}{3!}\alpha^3\beta B_1 - \frac{1}{5!}\alpha^5\beta B_1 + \frac{1}{7!}\alpha^7\beta B_2 - ... \\ &= \omega R + \beta(B_2\cos\alpha - B_1\sin\alpha), \end{aligned} \tag{3.44}$$

using (3.12) as well as the Maclaurin series of $\sin x$ and $\cos x$.

This tells us that a general linear combination of $\omega R + \beta^1 B_1 + \beta^2 B_2$ is physically equivalent to one with just $\omega R + \beta B_2$, so we do not need to consider B_1. We now have three cases, $\omega^2 > \beta^2$, $\omega^2 = \beta^2$ and $\omega^2 < \beta^2$.

3.3.1 Pure Rotation

Consider the next term:

$$\begin{aligned} e^{\chi B_1} R e^{-\chi B_1} &= R - \chi B_2 + \frac{1}{2!}\chi^2 R - \frac{1}{3!}\chi^3 B_2 + \frac{1}{4!}\chi^4 R - \frac{1}{5!}\chi^5 B_2 ... \\ &= R + \frac{1}{2!}\chi^2 R + \frac{1}{4!}\chi^4 R + ... \\ &\quad - \chi B_2 - \frac{1}{3!}\chi^3 B_2 - \frac{1}{5!}\chi^5 B_2 - ... \\ &= R\cosh\chi - B_2 \sinh\chi. \end{aligned} \tag{3.45}$$

In the case of $\omega^2 > \beta^2$, we can let $\tanh\chi = -\frac{\beta}{\omega}$ as we know that $-1 < \tanh\chi < 1$. This allows us to write

$$\begin{aligned} e^{\chi B_1} \frac{R}{\cosh\chi} e^{-\chi B_1} &= R - B_2 \tanh\chi \\ e^{\chi B_1} \frac{\omega R}{\cosh\chi} e^{-\chi B_1} &= \omega R + \beta B_2 \\ e^{\chi B_1} \omega\sqrt{1 - \frac{\beta^2}{\omega^2}} R e^{-\chi B_1} &= \omega R + \beta B_2, \end{aligned} \tag{3.46}$$

with the help of the hyperbolic trigonometric identities $\frac{\sinh\chi}{\cosh\chi} = \tanh\chi$ and $\frac{1}{\cosh\chi} = \sqrt{1 - \tanh^2\chi}$.

This allows us to identify flows with less translation than rotation as physically equivalent to a flow that is purely rotating. Given that

$$\begin{aligned} R = i(z\partial_z - \bar{z}\partial_{\bar{z}}) &= i \begin{bmatrix} \frac{\partial\theta}{\partial z} & \frac{\partial\theta}{\partial \bar{z}} \\ \frac{\partial\phi}{\partial z} & \frac{\partial\phi}{\partial \bar{z}} \end{bmatrix} \begin{bmatrix} z \\ -\bar{z} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \partial_\phi, \end{aligned} \tag{3.47}$$

the velocity profile of such a flow would be

$$u = \gamma(\partial_t + \omega\partial_\phi), \tag{3.48}$$

with

$$\gamma = \frac{1}{\sqrt{1 - \omega^2 r^2 \sinh^2\theta}}. \tag{3.49}$$

Speed of light considerations tell us that this fluid only exists at

$$\sinh\theta < \frac{1}{|\omega|\, r}. \tag{3.50}$$

This leads to a stress tensor of

$$\begin{aligned}
T^{\mu\nu} &= \mathcal{P}(3u^\mu u^\nu + g^{\mu\nu}) \\
&= \mathcal{P}\left(3\begin{bmatrix} \gamma^2 & 0 & \gamma^2\omega \\ 0 & 0 & 0 \\ \gamma^2\omega & 0 & \gamma^2\omega^2 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sinh^2\theta} \end{bmatrix}\right) \\
&= \mathcal{P}\begin{bmatrix} 3\gamma^2 - 1 & 0 & 3\gamma^2\omega \\ 0 & \frac{1}{r^2} & 0 \\ 3\gamma^2\omega & 0 & \frac{3\gamma^2-2}{r^2\sinh^2\theta} \end{bmatrix}.
\end{aligned} \tag{3.51}$$

3.3.2 Pure Translation

Consider this:

$$\begin{aligned}
e^{\chi B_1} B_2 e^{-\chi B_1} &= B_2 - \chi R + \frac{1}{2!}\chi^2 B_2 - \frac{1}{3!}\chi^3 R + \frac{1}{4!}\chi^4 B_2 - \frac{1}{5!}\chi^5 R ... \\
&= B_2 + \frac{1}{2!}\chi^2 B_2 + \frac{1}{4!}\chi^4 B_2 + ... \\
&\quad - \chi R - \frac{1}{3!}\chi^3 R - \frac{1}{5!}\chi^5 R - ... \\
&= B_2 \cosh\chi - R\sinh\chi.
\end{aligned} \tag{3.52}$$

When $\omega^2 < \beta^2$, we can set $\tanh\chi = -\frac{\omega}{\beta}$, giving us

$$\begin{aligned}
e^{\chi B_1}\frac{B_2}{\cosh\chi} e^{-\chi B_1} &= -R\tanh\chi + B_2 \\
e^{\chi B_1}\frac{\beta B_2}{\cosh\chi} e^{-\chi B_1} &= \omega R + \beta B_2 \\
e^{\chi B_1}\beta\sqrt{1 - \frac{\omega^2}{\beta^2}} B_2 e^{-\chi B_1} &= \omega R + \beta B_2,
\end{aligned} \tag{3.53}$$

which tells us that if the rotational flow magnitude is less than the translational flow, it is physically equivalent to that of one with pure translation.

To rewrite the flow in a more convenient manner, we have the coordinates

$$X = \sinh\theta\cos\phi, \quad Y = \sinh\theta\sin\phi. \tag{3.54}$$

The relevant information would be

$$\begin{aligned}
\phi = \tan^{-1}\frac{Y}{X}, \quad \theta = \sinh^{-1}\sqrt{Y^2+X^2}, \quad \sinh^2\theta = Y^2+X^2,\\
\frac{\partial\phi}{\partial X} = -\frac{Y}{X^2+Y^2}, \quad \frac{\partial\phi}{\partial Y} = \frac{X}{X^2+Y^2},\\
\frac{\partial\theta}{\partial X} = \frac{X}{\sqrt{(1+X^2+Y^2)(X^2+Y^2)}}, \quad \frac{\partial\theta}{\partial Y} = \frac{Y}{\sqrt{(1+X^2+Y^2)(X^2+Y^2)}}.
\end{aligned} \tag{3.55}$$

We now have the transformation

$$\begin{aligned}
\bar{g}_{ij} &= r^2 \begin{bmatrix} \frac{\partial\theta}{\partial X} & \frac{\partial\phi}{\partial X} \\ \frac{\partial\theta}{\partial Y} & \frac{\partial\phi}{\partial Y} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sinh^2\theta \end{bmatrix} \begin{bmatrix} \frac{\partial\theta}{\partial X} & \frac{\partial\theta}{\partial Y} \\ \frac{\partial\phi}{\partial X} & \frac{\partial\phi}{\partial Y} \end{bmatrix}\\
&= \frac{r^2}{1+X^2+Y^2} \begin{bmatrix} 1+Y^2 & -XY \\ -XY & 1+X^2 \end{bmatrix},
\end{aligned} \tag{3.56}$$

giving the metric

$$ds^2 = -dt^2 + \frac{r^2}{1+X^2+Y^2}\left((1+Y^2)dX^2 + (1+X^2)dY^2 - 2XY\,dX\,dY\right). \tag{3.57}$$

We now note that

$$\begin{aligned}
\frac{\partial X}{\partial\theta} = \cosh\theta\cos\phi, \quad \frac{\partial X}{\partial\phi} = -\sinh\theta\sin\phi,\\
\frac{\partial Y}{\partial\theta} = \cosh\theta\sin\phi, \quad \frac{\partial Y}{\partial\phi} = \sinh\theta\cos\phi,
\end{aligned} \tag{3.58}$$

which allows us to find

$$\begin{aligned}
B_1 &= \frac{i}{2}(1+z^2)\partial_z - \frac{i}{2}(1+\bar{z}^2)\partial_{\bar{z}}\\
&= \frac{i}{2}\begin{bmatrix} \frac{\partial\theta}{\partial z} & \frac{\partial\theta}{\partial\bar{z}} \\ \frac{\partial\phi}{\partial z} & \frac{\partial\phi}{\partial\bar{z}} \end{bmatrix} \begin{bmatrix} 1+z^2 \\ -1-\bar{z}^2 \end{bmatrix}\\
&= \begin{bmatrix} \sin\phi \\ \coth\theta\cos\phi \end{bmatrix} = \sin\phi\partial_\theta + \coth\theta\cos\phi\partial_\phi
\end{aligned} \tag{3.59}$$

$$\begin{aligned}
&= \begin{bmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \end{bmatrix} \begin{bmatrix} \sin\phi \\ \coth\theta\cos\phi \end{bmatrix}\\
&= \begin{bmatrix} 0 \\ \cosh\theta \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{1+X^2+Y^2} \end{bmatrix} = \sqrt{1+X^2+Y^2}\partial_Y,
\end{aligned} \tag{3.60}$$

$$\begin{aligned}
B_2 &= \frac{1}{2}(1-z^2)\partial_z + \frac{1}{2}(1-\bar{z}^2)\partial_{\bar{z}} \\
&= \frac{1}{2}\begin{bmatrix} \frac{\partial\theta}{\partial z} & \frac{\partial\theta}{\partial\bar{z}} \\ \frac{\partial\phi}{\partial z} & \frac{\partial\phi}{\partial\bar{z}} \end{bmatrix}\begin{bmatrix} 1-z^2 \\ 1-\bar{z}^2 \end{bmatrix} \\
&= \begin{bmatrix} \cos\phi \\ -\coth\theta\sin\phi \end{bmatrix} = \cos\phi\partial_\theta - \coth\theta\sin\phi\partial_\phi && (3.61) \\
&= \begin{bmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \end{bmatrix}\begin{bmatrix} \cos\phi \\ -\coth\theta\sin\phi \end{bmatrix} \\
&= \begin{bmatrix} \cosh\theta \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{1+X^2+Y^2} \\ 0 \end{bmatrix} = \sqrt{1+X^2+Y^2}\partial_X && (3.62)
\end{aligned}$$

and

$$R = [B_2, B_1] = -Y\partial_X + X\partial_Y. \tag{3.63}$$

We comment in brief that this coordinate system is symmetric between the 2 coordinates, and shares many similarities with the cartesian system, in terms of the flows. However, the metric contains off diagonal terms.

We thus have our purely translational flow of

$$u = \gamma\left(\partial_t + \beta\sqrt{1+X^2+Y^2}\partial_X\right), \tag{3.64}$$

with

$$\gamma = \frac{1}{\sqrt{1-\beta^2 r^2(1+Y^2)}}. \tag{3.65}$$

Note that the fluid will be bounded in the Y axis by

$$Y^2 < \frac{1}{\beta^2 r^2} - 1, \tag{3.66}$$

due to the speed of light. This also means that this flow only exists when

$$|\beta| < \frac{1}{r}. \tag{3.67}$$

The stress tensor is thus

$$
\begin{aligned}
T^{\mu\nu} &= \mathcal{P}\left(3u^\mu u^\nu + g^{\mu\nu}\right) \\
&= \mathcal{P}\left(3\begin{bmatrix} \gamma^2 & \gamma^2\beta\sqrt{1+X^2+Y^2} & 0 \\ \gamma^2\beta\sqrt{1+X^2+Y^2} & \gamma^2\beta^2\left(1+X^2+Y^2\right) & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1+X^2}{r^2} & \frac{XY}{r^2} \\ 0 & \frac{XY}{r^2} & \frac{1+Y^2}{r^2} \end{bmatrix}\right) \\
&= \mathcal{P}\begin{bmatrix} 3\gamma^2-1 & 3\gamma^2\beta\sqrt{1+X^2+Y^2} & 0 \\ 3\gamma^2\beta\sqrt{1+X^2+Y^2} & 3\gamma^2\beta^2\left(1+X^2+Y^2\right)+\frac{1+X^2}{r^2} & \frac{XY}{r^2} \\ 0 & \frac{XY}{r^2} & \frac{1+Y^2}{r^2} \end{bmatrix}.
\end{aligned}
\tag{3.68}
$$

3.3.3 Mixed flow

In the case of $\omega^2 = \beta^2$, we consider the transformation which swaps z and $\bar{z}$, resulting in

$$
R \rightarrow -R, \quad B_1 \rightarrow -B_1, \quad B_2 \rightarrow B_2, \tag{3.69}
$$

which means that the case of $\omega = -\beta$ is related by an isometry to the case of $\omega = \beta$ and are thus physically equivalent.

We now introduce again another set of coordinates to suit this flow better, with

$$
A = \ln\frac{1-z\bar{z}}{z\bar{z}+i(z-\bar{z})+1}, \quad B = \frac{z+\bar{z}}{z\bar{z}+i(z-\bar{z})+1}. \tag{3.70}
$$

We now have

$$
\begin{aligned}
\frac{\partial A}{\partial z} &= \frac{-\bar{z}(z\bar{z}+i(z-\bar{z})+1)-(1-z\bar{z})(\bar{z}+i)}{(1-z\bar{z})(z\bar{z}+i(z-\bar{z})+1)}, \\
\frac{\partial A}{\partial \bar{z}} &= \frac{-z(z\bar{z}+i(z-\bar{z})+1)-(1-z\bar{z})(z-i)}{(1-z\bar{z})(z\bar{z}+i(z-\bar{z})+1)}, \\
\frac{\partial B}{\partial z} = \frac{-\bar{z}^2-2i\bar{z}+1}{(z\bar{z}+i(z-\bar{z})+1)^2}, &\quad \frac{\partial B}{\partial \bar{z}} = \frac{-z^2+2iz+1}{(z\bar{z}+i(z-\bar{z})+1)^2}.
\end{aligned}
\tag{3.71}
$$

Transforming our flow to new coordinates, we have

$$\begin{aligned} R + B_2 &= (iz + \frac{1}{2}(1 - z^2))\partial_z + (-i\bar{z} + \frac{1}{2}(1 - \bar{z}^2))\partial_{\bar{z}} \\ &= \begin{bmatrix} \frac{\partial A}{\partial z} & \frac{\partial A}{\partial \bar{z}} \\ \frac{\partial B}{\partial z} & \frac{\partial B}{\partial \bar{z}} \end{bmatrix} \begin{bmatrix} iz + \frac{1}{2}(1 - z^2) \\ -i\bar{z} + \frac{1}{2}(1 - \bar{z}^2) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \partial_B, \end{aligned} \tag{3.72}$$

telling us that this mixed flow is purely in the B axis. We note that the metric can be written as

$$\bar{g}^{ij} = \frac{(1 - z\bar{z})^2}{2r^2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{3.73}$$

in the Poincaré disk coordinates by inverting (3.38). Changing coordinates, we have

$$\begin{aligned} \bar{g}^{ij} &= \frac{(1 - z\bar{z})^2}{2r^2} \begin{bmatrix} \frac{\partial A}{\partial z} & \frac{\partial A}{\partial \bar{z}} \\ \frac{\partial B}{\partial z} & \frac{\partial B}{\partial \bar{z}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial A}{\partial z} & \frac{\partial B}{\partial z} \\ \frac{\partial A}{\partial \bar{z}} & \frac{\partial B}{\partial \bar{z}} \end{bmatrix} \\ &= \frac{1}{r^2} \begin{bmatrix} 1 & 0 \\ 0 & e^{2A} \end{bmatrix}, \end{aligned} \tag{3.74}$$

giving us

$$\bar{g}_{ij} = r^2 \begin{bmatrix} 1 & 0 \\ 0 & e^{-2A} \end{bmatrix} \tag{3.75}$$

or

$$ds^2 = dt^2 + r^2(dA^2 + e^{-2A}dB^2). \tag{3.76}$$

Thus the velocity profile of the flow is

$$u = \gamma(\partial_t + \beta\partial_B), \tag{3.77}$$

with

$$\gamma = \frac{1}{\sqrt{1 - \beta^2 r^2 e^{-2A}}}. \tag{3.78}$$

The speed of light means that the flow only exists in regions where

$$e^{-A} < \frac{1}{|\beta|\, r}. \tag{3.79}$$

The stress tensor is given by

$$\begin{aligned}
T^{\mu\nu} &= \mathcal{P}\left(3u^{\mu}u^{\nu} + g^{\mu\nu}\right) \\
&= \mathcal{P}\left(3\begin{bmatrix} \gamma^2 & 0 & \gamma^2\beta \\ 0 & 0 & 0 \\ \gamma^2\beta & 0 & \gamma^2\beta^2 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{e^{2A}}{r^2} \end{bmatrix}\right) \\
&= \mathcal{P}\begin{bmatrix} 3\gamma^2 - 1 & 0 & 3\gamma^2\beta \\ 0 & \frac{1}{r^2} & 0 \\ 3\gamma^2\beta & 0 & 3\gamma^2\beta^2 + \frac{e^{2A}}{r^2} \end{bmatrix}.
\end{aligned} \tag{3.80}$$

Chapter 4

The Carter–Plebański Class of Black Holes

The Carter–Plebański class of black holes [13, 14] is given by the metric

$$ds^2 = \frac{p^2 + q^2}{P(p)} dp^2 + \frac{P(p)}{p^2 + q^2}(d\tau + q^2 d\sigma)^2 + \frac{p^2 + q^2}{Q(q)} dq^2 - \frac{Q(q)}{p^2 + q^2}(d\tau - p^2 d\sigma)^2 \tag{4.1}$$

with

$$\begin{aligned} P(p) &= k + 2np - \epsilon p^2 - \frac{\Lambda}{3} p^4, \\ Q(q) &= k + e^2 + g^2 - 2mq + \epsilon q^2 - \frac{\Lambda}{3} q^4, \end{aligned} \tag{4.2}$$

where e is the electric charge, g is the magnetic charge, while Λ is the cosmological constant. The rest of the parameters gain different physical meaning in different situations.

This class is a general class of black holes, and contains the Kerr–Newman–NUT–(Anti-)de Sitter class of black holes in it. This class can be obtained from the the most general class of black holes, known as the Plebański–Demiański class [15], via a scaling limit by taking the acceleration to zero. As we are dealing with Anti-de Sitter space (negative cosmological constant), we can do the replacement

$$\Lambda = -\frac{3}{\ell^2}, \tag{4.3}$$

with ℓ taken to be a positive quantity.

The boundary of the metric would be given by taking $q \to \infty$ as well as making it constant, and dropping off lower order terms (with respect to q). Noting that under these conditions

$$\frac{Q(q)}{q^2} = \frac{q^2}{\ell^2} + \epsilon, \tag{4.4}$$

the boundary metric is

$$\begin{aligned} d\check{s}^2 &= \frac{q^2}{P(p)}dp^2 + \frac{P(p)}{q^2}(d\tau + q^2 d\sigma)^2 - \frac{Q(q)}{q^2}(d\tau - p^2 d\sigma)^2 \\ &= -\frac{Q(q)}{q^2}d\tau^2 + \frac{q^2}{P(p)}dp^2 + (q^2P(p) - \frac{Q(q)}{q^2}p^4)d\sigma^2 + 2(P(p) + \frac{Q(q)}{q^2}p^2)d\tau d\sigma \\ &= \frac{q^2}{\ell^2}(-d\tau^2 + \frac{\ell^2}{P(p)}dp^2 + (\ell^2 P(p) - p^4)d\sigma^2 + 2p^2 d\tau d\sigma). \end{aligned} \tag{4.5}$$

Performing a conformal transformation $\hat{g}^{\mu\nu} = \frac{\ell^2}{q^2}\check{g}^{\mu\nu}$, the boundary metric becomes

$$d\hat{s}^2 = -d\tau^2 + \frac{\ell^2}{P(p)}dp^2 + (\ell^2 P(p) - p^4)d\sigma^2 + 2p^2 d\tau d\sigma. \tag{4.6}$$

The Cotton tensor for this metric vanishes when $n = 0$. As the ultrastatic metrics with constant curvature in the spatial sections are conformally flat (i.e. Cotton tensor vanishes), we will only consider $n = 0$. Also, we will ignore charge, as a charged fluid will have a net Lorentz force of zero and thus would not behave any differently from that of an uncharged one [6]. Thus, for the rest of this thesis,

$$n = e = g = 0. \tag{4.7}$$

The holographic stress tensor [16] dual to the bulk containing the black hole [6] is

$$\hat{T}^{\mu\nu} = \frac{m}{8\pi\ell^2}(3u^\mu u^\nu + \hat{g}^{\mu\nu}), \tag{4.8}$$

with

$$u^\mu = \partial_\tau, \tag{4.9}$$

i.e. an stationary fluid on the boundary.

For the black holes dual to the fluid flows classified earlier, an appropriate additional coordinate and conformal transformation of the boundary would be required to obtain the necessary ultrastatic coordinates. This will be detailed later on in this chapter.

4.1 A Classification via the quartic $P(p)$

$P(p)$ is a very important quartic in the metric. $P(p)$ needs to be positive for the metric to be of the right signature, and the location and number of real roots determines the shape of the horizon.

After setting $n = 0$ as stated in the previous section, we have

$$P(p) = k - \epsilon p^2 + \frac{p^4}{\ell^2}, \tag{4.10}$$

which is a quadratic in terms of p^2. Thus, if the discriminant

$$\Delta = \epsilon^2 - \frac{4k}{\ell^2} \tag{4.11}$$

is bigger or equal to zero, $\Delta \geq 0$, the quartic can be rewritten as

$$P(p) = \frac{1}{\ell^2}(p^2 - \alpha_+)(p^2 - \alpha_-), \tag{4.12}$$

with

$$\alpha_\pm = \frac{\ell^2}{2}(\epsilon \pm \sqrt{\Delta}). \tag{4.13}$$

We will now consider a classification of black holes via the various cases of $P(p)$. When the black holes are dual to the fluid flows classified in the previous chapter, they will be discussed in depth.

4.1.1 Case 1: $k > 0,\ \epsilon > 2\sqrt{k}/\ell$

When

$$k > 0, \quad \epsilon > 2\sqrt{k}/\ell, \tag{4.14}$$

then

$$\Delta > 0, \quad \sqrt{\Delta} < \epsilon, \quad \alpha_\pm > 0, \tag{4.15}$$

which means that $P(p)$ has four real roots, with

$$P(p) = \frac{1}{\ell^2}(p - \sqrt{\alpha_+})(p + \sqrt{\alpha_+})(p - \sqrt{\alpha_-})(p + \sqrt{\alpha_-}). \tag{4.16}$$

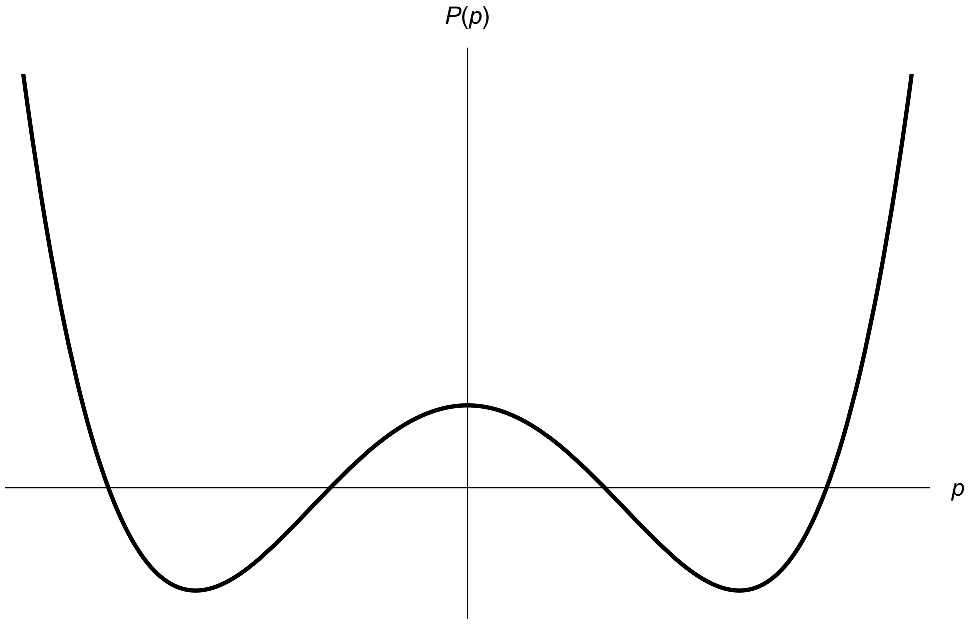

FIGURE 4.1: Plot of $P(p)$ for case 1. The Kerr black hole lives in the centre region where p is positive, with the roots corresponding to the poles marking the axis that the black hole rotates about. The black hole dual to rotating flow on the polar regions lives in the region on the right where p is positive if $a > \ell$, and on the left if $a < -\ell$.

Now $P(p)$ is positive for two ranges,

$$|p| > \sqrt{\alpha_+}, \quad |p| < \sqrt{\alpha_-}. \tag{4.17}$$

4.1.1.1 Black hole dual to rotational fluid flow over the entire 2-sphere (the Kerr–AdS black hole)

For the region $|p| < \sqrt{\alpha_-}$, this actually corresponds to the spherical Kerr–AdS black hole. We see this by rescaling to set $\alpha_+ = \ell^2$ and defining $a^2 := \alpha_-$.

Taking the metric (4.1) and setting

$$k = a^2, \quad \epsilon = 1 + \frac{a^2}{\ell^2}, \quad \tau = t - a\phi, \quad q = r, \quad p = a\cos\theta, \quad \sigma = -\frac{\phi}{a}, \tag{4.18}$$

giving the metric

$$ds^2 = -\frac{\Delta_r}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \rho^2\left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right) + \frac{\Delta_\theta \sin^2\theta}{\rho^2}(adt - (r^2 + a^2)d\phi)^2, \tag{4.19}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta_r = (r^2+a^2)\left(1+\frac{r^2}{\ell^2}\right) - 2mr, \quad \Delta_\theta = 1 - \frac{a^2}{\ell^2}\cos^2\theta. \tag{4.20}$$

This is simply the Kerr–AdS black hole [17]. We now extract the boundary metric,

$$\begin{aligned} d\check{s}^2 &= \frac{\Delta_r}{\rho^2}\Bigg(\left(-1+\frac{\Delta_\theta\sin^2\theta}{\Delta_r}\right)dt^2 + \frac{\rho^4}{\Delta_r\Delta_\theta}d\theta^2 + \left(\frac{\Delta_\theta\sin^2\theta}{\Delta_r}(r^2+a^2)^2 - a^2\sin^4\theta\right)d\phi^2 \\ &\quad + \left(2a\sin^2\theta - \frac{\Delta_\theta\sin^2\theta}{\Delta_r}2a(r^2+a^2)\right)dtd\phi\Bigg) \\ &= \frac{\Delta_r}{\rho^2}\left(-dt^2 + \frac{\ell^2}{\Delta_\theta}d\theta^2 + \sin^2\theta\left(\ell^2\Delta_\theta - a^2\sin^2\theta\right)d\phi^2 + 2a\sin^2\theta dtd\phi\right) \\ &= \frac{\Delta_r}{\rho^2}\left(-dt^2 + \frac{\ell^2}{\Delta_\theta}d\theta^2 + \sin^2\theta\left(\ell^2 - a^2\right)d\phi^2 + 2a\sin^2\theta dtd\phi\right). \end{aligned} \tag{4.21}$$

After a conformal transformation,

$$d\hat{s}^2 = -dt^2 + \frac{\ell^2}{\Delta_\theta}d\theta^2 + \sin^2\theta\left(\ell^2 - a^2\right)d\phi^2 + 2a\sin^2\theta dtd\phi, \tag{4.22}$$

where we note that it can be obtained directly from (4.6) after setting (4.18).

We now transform via the following coordinates,

$$\cos\Theta = \cos\theta\sqrt{\frac{\Xi}{\Delta_\theta}}, \quad \Phi = \Xi\phi + \frac{at}{\ell^2}, \tag{4.23}$$

where

$$\Xi = 1 - \frac{a^2}{\ell^2}, \tag{4.24}$$

yielding

$$d\hat{s}^2 = \frac{\Delta_\theta}{\Xi}\left(-dt^2 + \ell^2(d\Theta^2 + \sin^2\Theta d\Phi^2)\right), \tag{4.25}$$

which is a 2-sphere up to a conformal transformation. Note that the range of Θ is $0 \leq \Theta \leq \pi$, which means the metric covers the whole sphere.

Performing a conformal transformation of

$$\tilde{g}^{\mu\nu} = \Omega^2\hat{g}^{\mu\nu} = \frac{\Xi}{\Delta_\theta}\hat{g}^{\mu\nu}, \tag{4.26}$$

we have

$$\tilde{ds}^2 = -dt^2 + \ell^2(d\Theta^2 + \sin^2\Theta d\Phi^2). \tag{4.27}$$

We remember that the fluid on the boundary is stationary with respect to the initial coordinates (4.9), so transforming to the black hole coordinates,

$$\begin{aligned} u &= \partial_\tau \\ &= \begin{bmatrix} \frac{\partial t}{\partial \tau} & \frac{\partial t}{\partial p} & \frac{\partial t}{\partial \sigma} \\ \frac{\partial \theta}{\partial \tau} & \frac{\partial \theta}{\partial p} & \frac{\partial \theta}{\partial \sigma} \\ \frac{\partial \phi}{\partial \tau} & \frac{\partial \phi}{\partial p} & \frac{\partial \phi}{\partial \sigma} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \partial_t, \end{aligned} \tag{4.28}$$

noting that $\frac{\partial t}{\partial \tau} = 1$, $\frac{\partial \theta}{\partial \tau} = 0$ and $\frac{\partial \phi}{\partial \tau} = 0$, which tells us that the fluid is still stationary in these coordinates.

The associated stress tensor to (4.22) would thus be

$$\hat{T}^{\mu\nu} = \frac{m}{8\pi\ell^2}(\hat{g}^{\mu\nu} + 3u^\mu u^\nu), \tag{4.29}$$

with $u = \partial_t$.

The fluid now transforms via (4.23) and (4.26) into

$$\tilde{u} = \Omega^{-1}u = \Omega^{-1}\left(\partial_t + \frac{a}{\ell^2}\partial_\Phi\right) = \gamma\left(\partial_t + \frac{a}{\ell^2}\partial_\Phi\right), \tag{4.30}$$

where we notice that

$$\Omega^{-1} = \sqrt{\frac{\Delta_\theta}{\Xi}} = \frac{1}{\sqrt{1 - \frac{a^2}{\ell^2}\sin^2\Theta}} = \gamma. \tag{4.31}$$

We obtain the stress tensor associated with (4.27),

$$\tilde{T}^{\mu\nu} = \Omega^{-5}\hat{T}^{\mu\nu} = \frac{m\gamma^3}{8\pi\ell^2}(\tilde{g}^{\mu\nu} + 3\tilde{u}^\mu\tilde{u}^\nu), \tag{4.32}$$

where we make the identification of

$$\mathcal{P} = \gamma^3 \mathcal{P}_0 = \frac{m\gamma^3}{8\pi\ell^2}, \quad \omega = \frac{a}{\ell^2}, \quad r = \ell, \tag{4.33}$$

mapping it to that of the fluid flow on the 2-sphere, as seen in (3.31). This is a well known result, as seen in [4].

We now note that $a^2 < \ell^2$, which is needed to maintain the signature of the black hole metric (4.19) by keeping $\Delta_\theta > 0$, or alternatively, because of the rescaling of the quartic $P(p)$ as mentioned at the start of this section. This requirement means that the fluid flow is limited to

$$|\omega| < \frac{1}{r}, \tag{4.34}$$

which allows the fluid to cover the whole sphere, as shown in (3.32).

4.1.1.2 Black hole dual to rotational flow on the polar region of the 2-sphere

In the region $|p| > \sqrt{\alpha_+}$, we set $\alpha_- = \ell^2$ and define $a^2 := \alpha_+$.

From the original metric (4.1) we now set

$$k = a^2, \quad \epsilon = 1 + \frac{a^2}{\ell^2}, \quad \tau = t - a\phi, \quad q = r, \quad p = a\cosh\theta, \quad \sigma = -\frac{\phi}{a}, \tag{4.35}$$

yielding the metric

$$ds^2 = -\frac{\Delta_r}{\rho^2}(dt - a\sinh^2\theta d\phi)^2 + \rho^2\left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right) + \frac{\Delta_\theta \sinh^2\theta}{\rho^2}(adt - (r^2+a^2)d\phi)^2, \tag{4.36}$$

where

$$\rho^2 = r^2 + a^2\cosh^2\theta, \quad \Delta_r = (r^2+a^2)\left(1 + \frac{r^2}{\ell^2}\right) - 2mr, \quad \Delta_\theta = \frac{a^2}{\ell^2}\cosh^2\theta - 1. \tag{4.37}$$

We note that this metric can also be obtained by the analytic continuation of $\theta \to i\theta$ applied on the Kerr–AdS metric (4.19). Also, to maintain the signature of this black hole, we need $a^2 > \ell^2$ to maintain the positivity of Δ_θ, as $\cosh\theta \geq 1$.

Extracting the boundary metric, we now have

$$d\check{s}^2 = \frac{\Delta_r}{\rho^2}\left(-dt^2 + \frac{\ell^2}{\Delta_\theta}d\theta^2 + \sinh^2\theta\left(a^2 - \ell^2\right)d\phi^2 + 2a\sinh^2\theta dt d\phi\right), \tag{4.38}$$

and then we apply the first conformal transformation,

$$d\check{s}^2 = -dt^2 + \frac{\ell^2}{\Delta_\theta} d\theta^2 + \sinh^2\theta \left(a^2 - \ell^2\right) d\phi^2 + 2a \sinh^2\theta dt d\phi. \tag{4.39}$$

With a new set of coordinates,

$$\sin\Theta = \frac{\sinh\theta}{\sqrt{\Delta_\theta}}, \quad \Phi = \Xi\phi - \frac{at}{\ell^2}, \tag{4.40}$$

where

$$\Xi = \frac{a^2}{\ell^2} - 1, \tag{4.41}$$

the metric is now

$$d\hat{s}^2 = \frac{\Delta_\theta}{\Xi}\left(-dt^2 + \ell^2(d\Theta^2 + \sin^2\Theta d\Phi^2)\right). \tag{4.42}$$

Note that

$$\frac{d}{d\theta}\sin\Theta = \frac{(\frac{a^2}{\ell^2} - 1)\cosh(\theta)}{\left(\frac{a^2}{\ell^2}\cosh^2(\theta) - 1\right)^{3/2}} \tag{4.43}$$

and

$$\lim_{\theta\to+\infty} \frac{\sinh\theta}{\sqrt{\Delta_\theta}} = \lim_{\theta\to+\infty} \frac{\ell}{|a|}\tanh\theta = \frac{\ell}{|a|}, \tag{4.44}$$

which means $\sin\Theta$ is strictly increasing for $\theta \geq 0$, and that it limits to $\frac{\ell}{|a|}$, making the range of Θ to be

$$0 \leq \Theta < \sin^{-1}\frac{\ell}{|a|}. \tag{4.45}$$

Performing a conformal transformation of

$$\tilde{g}^{\mu\nu} = \Omega^2 \hat{g}^{\mu\nu} = \frac{\Xi}{\Delta_\theta}\hat{g}^{\mu\nu}, \tag{4.46}$$

we now have

$$\tilde{ds^2} = -dt^2 + \ell^2(d\Theta^2 + \sin^2\Theta d\Phi^2), \tag{4.47}$$

which is again the 2-sphere, but this time only to $\Theta < \sin^{-1}\frac{\ell}{|a|}$.

The associated stress tensor to (4.39) would be

$$\hat{T}^{\mu\nu} = \frac{m}{8\pi\ell^2}(\hat{g}^{\mu\nu} + 3u^\mu u^\nu), \tag{4.48}$$

with $u = \partial_t$, remembering (4.28).

The fluid transforms via (4.40) and (4.46) into

$$\tilde{u} = \Omega^{-1} u = \Omega^{-1} \left(\partial_t - \frac{a}{\ell^2} \partial_\Phi \right) = \gamma \left(\partial_t - \frac{a}{\ell^2} \partial_\Phi \right), \tag{4.49}$$

where we notice yet again that

$$\Omega^{-1} = \sqrt{\frac{\Delta_\theta}{\Xi}} = \frac{1}{\sqrt{1 - \frac{a^2}{\ell^2} \sin^2 \Theta}} = \gamma. \tag{4.50}$$

We now obtain the stress tensor associated with (4.47),

$$\tilde{T}^{\mu\nu} = \Omega^{-5} \hat{T}^{\mu\nu} = \frac{m\gamma^3}{8\pi\ell^2} (\tilde{g}^{\mu\nu} + 3\tilde{u}^\mu \tilde{u}^\nu), \tag{4.51}$$

where we make the identification of

$$\mathcal{P} = \gamma^3 \mathcal{P}_0 = \frac{m\gamma^3}{8\pi\ell^2}, \quad \omega = -\frac{a}{\ell^2}, \quad r = \ell. \tag{4.52}$$

As $a^2 > \ell^2$, we have

$$|\omega| > \frac{1}{r}. \tag{4.53}$$

We have thus found the dual to the fluid flows along the polar regions on the 2-sphere (3.31), that due to speed of light considerations, only flow till $\sin\theta < \frac{1}{|\omega| r}$. We note that the constraint on the initial black hole metric (4.36), $a^2 > \ell^2$, manifests itself as the boundary of the conformal boundary as well as corresponding exactly to this region in which the fluid flow does not exceed the speed of light (3.32).

4.1.2 Case 2: $k > 0,\ \epsilon = 2\frac{\sqrt{k}}{\ell}$

When

$$k > 0, \quad \epsilon = 2\frac{\sqrt{k}}{\ell}, \tag{4.54}$$

then

$$\Delta = 0, \quad \alpha_\pm = \ell\sqrt{k}, \tag{4.55}$$

giving

$$P(p) = \frac{1}{\ell^2} \left(p^2 - \alpha_\pm \right)^2 = \frac{1}{\ell^2} \left((p + \sqrt{\alpha_\pm})(p - \sqrt{\alpha_\pm}) \right)^2. \tag{4.56}$$

$P(p)$ is now positive for all p except $p \neq \sqrt{\alpha_\pm} = \sqrt{\ell\sqrt{k}}$.

4.1.3 Case 3: $k > 0, \ -2\frac{\sqrt{k}}{\ell} < \epsilon < 2\frac{\sqrt{k}}{\ell}$

When

$$k > 0, \quad -2\frac{\sqrt{k}}{\ell} < \epsilon < 2\frac{\sqrt{k}}{\ell}, \tag{4.57}$$

then

$$\Delta < 0, \tag{4.58}$$

which implies that $P(p) > 0$ always.

4.1.4 Case 4: $k > 0, \ \epsilon = -2\frac{\sqrt{k}}{\ell}$

When

$$k > 0, \quad \epsilon = -2\frac{\sqrt{k}}{\ell}, \tag{4.59}$$

then

$$\Delta = 0, \quad \alpha_\pm = -\ell\sqrt{k}, \tag{4.60}$$

giving

$$P(p) = \frac{1}{\ell^2}\left(p^2 + \ell\sqrt{k}\right)^2, \tag{4.61}$$

which is always positive.

4.1.5 Case 5: $k > 0, \ \epsilon < -2\frac{\sqrt{k}}{\ell}$

When

$$k > 0, \quad \epsilon < -2\frac{\sqrt{k}}{\ell}, \tag{4.62}$$

then

$$\Delta > 0, \quad \epsilon < -\sqrt{\Delta}, \quad \alpha_- < \alpha_+ < 0, \tag{4.63}$$

giving

$$P(p) = \frac{1}{\ell^2}(p^2 - \alpha_+)(p^2 - \alpha_-), \tag{4.64}$$

which has no real roots and is always positive.

4.1.5.1 Black hole dual to translational flow on the 2-hyperbola

We can rescale to $\epsilon = -1 - \frac{k}{\ell^2}$ and define $b^2 := k$. Taking the metric (4.1) and setting,

$$k = b^2, \quad \epsilon = -1 - \frac{k}{\ell^2}, \quad \tau = t + b\phi, \quad q = r, \quad p = b\sinh\theta, \quad \sigma = -\frac{\phi}{b}, \tag{4.65}$$

we gain the metric

$$ds^2 = -\frac{\Delta_r}{\rho^2}(dt + b\cosh^2\theta d\phi)^2 + \rho^2\left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right) + \frac{\Delta_\theta \cosh^2\theta}{\rho^2}(bdt - (r^2 - b^2)d\phi)^2, \quad (4.66)$$

where

$$\rho^2 = r^2 + b^2\sinh^2\theta, \quad \Delta_r = (r^2 - b^2)\left(1 + \frac{r^2}{\ell^2}\right) - 2mr, \quad \Delta_\theta = 1 + \frac{b^2}{\ell^2}\sinh^2\theta. \quad (4.67)$$

Extracting the boundary metric, we obtain

$$d\check{s}^2 = \frac{\Delta_r}{\rho^2}\left(-dt^2 + \frac{\ell^2}{\Delta_\theta}d\theta^2 + \cosh^2\theta\left(\ell^2 - b^2\right)d\phi^2 - 2b\cosh^2\theta dtd\phi\right), \quad (4.68)$$

giving the conformal boundary metric

$$d\hat{s}^2 = -dt^2 + \frac{\ell^2}{\Delta_\theta}d\theta^2 + \cosh^2\theta\left(\ell^2 - b^2\right)d\phi^2 - 2b\cosh^2\theta dtd\phi, \quad (4.69)$$

where we see that $\ell^2 > b^2$ is needed to maintain the correct (boundary) metric signature.

With the coordinates

$$X = \frac{\cosh\theta}{\sqrt{\Delta_\theta}}\sinh\left(\Xi\phi - \frac{bt}{\ell^2}\right), \quad Y = \sqrt{\frac{\Xi}{\Delta_\theta}}\sinh\theta, \quad (4.70)$$

where

$$\Xi = 1 - \frac{b^2}{\ell^2} \quad (4.71)$$

yields

$$d\hat{s}^2 = \frac{\Delta_\theta}{\Xi}\left(-dt^2 + \frac{\ell^2}{1 + X^2 + Y^2}((1 + Y^2)dX^2 + (1 + X^2)dY^2 - 2XYdXdY)\right). \quad (4.72)$$

Notice that

$$\frac{d}{d\theta}Y = 4\sqrt{4 - b^2}\cosh(\theta)\left(\frac{1}{b^2\sinh^2(\theta) + 4}\right)^{3/2}, \quad (4.73)$$

and

$$Y^2 = \frac{\Xi}{\Delta_\theta}\sinh^2\theta = \frac{1 - \frac{b^2}{\ell^2}}{\frac{1}{\sinh^2\theta} + \frac{b^2}{\ell^2}} \quad \to \quad \lim_{\theta\to\pm\infty} Y^2 = \frac{\ell^2}{b^2} - 1, \quad (4.74)$$

which means that the metric only exists for

$$Y^2 < \frac{\ell^2}{b^2} - 1. \quad (4.75)$$

Performing our conformal transformation of

$$\tilde{g}^{\mu\nu} = \Omega^2 \hat{g}^{\mu\nu} = \frac{\Xi}{\Delta_\theta} \hat{g}^{\mu\nu}, \tag{4.76}$$

we get the metric we desire,

$$\tilde{ds}^2 = -dt^2 + \frac{\ell^2}{1+X^2+Y^2} \left((1+Y^2)dX^2 + (1+X^2)dY^2 - 2XY dX dY \right), \tag{4.77}$$

which is the manifold of the 2-hyperbola, for the region of $Y^2 < \frac{\ell^2}{b^2} - 1$.

Similarly, the holographic stress tensor of (4.69) is

$$\hat{T}^{\mu\nu} = \frac{m}{8\pi\ell^2} (\hat{g}^{\mu\nu} + 3u^\mu u^\nu), \tag{4.78}$$

with $u = \partial_t$, where the results of (4.28) still holds for this black hole coordinates, which becomes

$$\tilde{T}^{\mu\nu} = \Omega^{-5} \hat{T}^{\mu\nu} = \frac{m\gamma^3}{8\pi\ell^2} (\tilde{g}^{\mu\nu} + 3\tilde{u}^\mu \tilde{u}^\nu), \tag{4.79}$$

with

$$\tilde{u} = \Omega^{-1} u = \gamma \left(\partial_t - \frac{b}{\ell^2} \sqrt{1+X^2+Y^2} \partial_X \right). \tag{4.80}$$

We now identify

$$\mathcal{P} = \gamma^3 \mathcal{P}_0 = \frac{m\gamma^3}{8\pi\ell^2}, \quad \gamma = \Omega^{-1} = \sqrt{\frac{\Delta_\theta}{\Xi}} = \frac{1}{\sqrt{1 - \frac{b^2}{\ell^2}(1+Y^2)}}, \quad \beta = -\frac{b}{\ell^2}, \quad r = \ell, \tag{4.81}$$

making this black hole the dual of the purely translating fluid flow in the 2-hyperbola (3.68). Yet again, the region in which the conformal boundary exists matches exactly to where the fluid exists, which is based on speed of light considerations (3.66).

4.1.6 Case 6: $k = 0, \ \epsilon > 0$

When

$$k = 0, \quad \epsilon > 0, \tag{4.82}$$

then

$$\Delta > 0, \quad \alpha_+ = \ell^2 \epsilon, \quad \alpha_- = 0 \tag{4.83}$$

giving

$$P(p) = \frac{p^2}{\ell^2}(p^2 - \ell^2\epsilon) = \frac{p^2}{\ell^2}(p - \ell\sqrt{\epsilon})(p + \ell\sqrt{\epsilon}), \tag{4.84}$$

which is positive for $|p| > \ell\sqrt{\epsilon}$.

4.1.6.1 Black hole dual to rotational flow on the Euclidean plane

We define $a^2 := \ell^2\epsilon$. Now setting the following on the metric (4.1),

$$k = 0, \quad \epsilon = \frac{a^2}{\ell^2}, \quad q = r, \quad p = aP, \quad \sigma = -\frac{\phi}{a}, \tag{4.85}$$

we obtain the metric

$$ds^2 = -\frac{\Delta_r}{\rho^2}(d\tau + aP^2 d\phi)^2 + \rho^2\left(\frac{dr^2}{\Delta_r} + \frac{dP^2}{\Delta_P}\right) + \frac{\Delta_P}{\rho^2}(ad\tau - r^2 d\phi)^2, \tag{4.86}$$

where

$$\rho^2 = r^2 + a^2P^2, \quad \Delta_r = (r^2 + a^2)\frac{r^2}{\ell^2} - 2mr, \quad \Delta_P = \frac{a^2}{\ell^2}P^2(P^2 - 1). \tag{4.87}$$

The boundary metric is

$$d\check{s}^2 = \frac{\Delta_r}{\rho^2}\left(-d\tau^2 + \frac{\ell^2}{\Delta_P}dP^2 - a^2P^2 d\phi^2 - 2aP^2 d\tau d\phi\right), \tag{4.88}$$

and the conformal boundary metric is

$$d\hat{s}^2 = -d\tau^2 + \frac{\ell^2}{\Delta_P}dP^2 - a^2P^2 d\phi^2 - 2aP^2 d\tau d\phi, \tag{4.89}$$

where we note that ∂_ϕ is actually timelike on the boundary. Another point of note is that $P^2 > 1$ to maintain the signature of the metric, so the appropriate range is $P > 1$.

The new coordinates are given by

$$T = \tau + a\phi, \quad \Phi = \frac{a\tau}{\ell^2}, \quad R = \frac{\ell^2}{a}\sqrt{1 - P^{-2}}, \tag{4.90}$$

providing us with the metric

$$d\hat{s}^2 = P^2\left(-dT^2 + dR^2 + R^2 d\Phi^2\right), \tag{4.91}$$

which after the conformal transformation of

$$\tilde{g}^{\mu\nu} = \Omega^2 \hat{g}^{\mu\nu} = P^{-2}\hat{g}^{\mu\nu}, \tag{4.92}$$

leads to

$$\tilde{ds}^2 = -dT^2 + dR^2 + R^2 d\Phi^2. \tag{4.93}$$

As $P > 1$,

$$R = \frac{\ell^2}{a}\sqrt{1 - P^{-2}} \quad \rightarrow \quad 0 < R < \frac{\ell^2}{a}. \tag{4.94}$$

We now have the Euclidean plane, in the region $0 < R < \frac{\ell^2}{a}$.

The stress tensor of (4.89) is the standard

$$\hat{T}^{\mu\nu} = \frac{m}{8\pi\ell^2}(\hat{g}^{\mu\nu} + 3u^\mu u^\nu), \tag{4.95}$$

with $u = \partial_\tau$, which becomes

$$\tilde{T}^{\mu\nu} = \Omega^{-5}\hat{T}^{\mu\nu} = \frac{m\gamma^3}{8\pi\ell^2}(\tilde{g}^{\mu\nu} + 3\tilde{u}^\mu \tilde{u}^\nu), \tag{4.96}$$

with

$$\tilde{u} = \Omega^{-1}u = \gamma(\partial_t + \frac{a}{\ell^2}\partial_\Phi). \tag{4.97}$$

Identifying

$$\mathcal{P} = \gamma^3 \mathcal{P}_0 = \frac{m\gamma^3}{8\pi\ell^2}, \quad \gamma = \Omega^{-1} = P = \frac{1}{\sqrt{1 - \frac{a^2 R^2}{\ell^4}}}, \quad \omega = \frac{a}{\ell^2}, \tag{4.98}$$

we have now shown that this black hole is the dual to pure rotational flow on the Euclidean plane (3.22), for the region where such a flow exists (3.21).

4.1.7 Case 7: $k = 0,\ \epsilon = 0$

When

$$k = 0, \quad \epsilon = 0, \tag{4.99}$$

then

$$\Delta = 0, \quad \alpha_\pm = 0, \tag{4.100}$$

giving

$$P(p) = \frac{p^4}{\ell^2}, \tag{4.101}$$

which is always positive except for $p = 0$.

4.1.8 Case 8: $k = 0,\ \epsilon < 0$

When

$$k = 0, \quad \epsilon < 0, \tag{4.102}$$

then

$$\Delta > 0, \quad \alpha_+ = 0, \quad \alpha_- = \ell^2\epsilon \tag{4.103}$$

giving

$$P(p) = \frac{p^2}{\ell^2}(p^2 - \ell^2\epsilon), \tag{4.104}$$

which is always positive except for $p = 0$.

4.1.8.1 Black hole dual to mixed flow on the 2-hyperbola

For the above case, we can rescale to get $\epsilon = -1$. Taking the metric (4.1) and setting

$$k = 0, \quad \epsilon = -1, \quad q = r, \quad p = aP, \quad \sigma = -\frac{\phi}{a}, \tag{4.105}$$

we acquire the metric

$$ds^2 = -\frac{\Delta_r}{\rho^2}(d\tau + aP^2 d\phi)^2 + \rho^2\left(\frac{dr^2}{\Delta_r} + \frac{dP^2}{\Delta_P}\right) + \frac{\Delta_P}{\rho^2}(ad\tau - r^2 d\phi)^2, \tag{4.106}$$

where

$$\rho^2 = r^2 + a^2P^2, \quad \Delta_r = r^2\left(\frac{r^2}{\ell^2} - 1\right) - 2mr, \quad \Delta_P = P^2(1 - \frac{a^2}{\ell^2}P^2), \tag{4.107}$$

with $P > 0$. The boundary metric is

$$d\check{s}^2 = \frac{\Delta_r}{\rho^2}\left(-d\tau^2 + \frac{\ell^2}{\Delta_P}dP^2 + \ell^2P^2d\phi^2 - 2aP^2 d\tau d\phi\right), \tag{4.108}$$

and the conformal boundary metric is

$$d\hat{s}^2 = -d\tau^2 + \frac{\ell^2}{\Delta_P}dP^2 + \ell^2P^2d\phi^2 - 2aP^2 d\tau d\phi. \tag{4.109}$$

Introducing the coordinates,

$$A = \frac{1}{2}\ln\frac{\Delta_P}{P^4}, \quad B = \phi - \frac{a\tau}{\ell^2}, \tag{4.110}$$

the metric can be rewritten as

$$d\hat{s}^2 = \frac{\Delta_P}{P^2}\left(-d\tau^2 + \ell^2(dA^2 + e^{-2A}dB^2)\right), \tag{4.111}$$

where a conformal transformation of

$$\tilde{g}^{\mu\nu} = \Omega^2 \hat{g}^{\mu\nu} = \frac{P^2}{\Delta_P}\hat{g}^{\mu\nu}, \tag{4.112}$$

yields

$$\tilde{ds}^2 = -d\tau^2 + \ell^2(dA^2 + e^{-2A}dB^2). \tag{4.113}$$

We see that

$$A = \frac{1}{2}\ln\left(\frac{1}{P^2} + \frac{a^2}{\ell^2}\right) \quad \rightarrow \quad A > \frac{1}{2}\ln\frac{a^2}{\ell^2} \quad \rightarrow \quad e^{-A} > \frac{\ell}{|a|}, \tag{4.114}$$

which implies that this metric only exists on part of the 2-hyperbola.

The stress tensor of (4.109) is the standard

$$\hat{T}^{\mu\nu} = \frac{m}{8\pi\ell^2}(\hat{g}^{\mu\nu} + 3u^\mu u^\nu), \tag{4.115}$$

with $u = \partial_\tau$, which becomes

$$\tilde{T}^{\mu\nu} = \Omega^{-5}\hat{T}^{\mu\nu} = \frac{m\gamma^3}{8\pi\ell^2}(\tilde{g}^{\mu\nu} + 3\tilde{u}^\mu\tilde{u}^\nu), \tag{4.116}$$

with

$$\tilde{u} = \Omega^{-1}u = \gamma(\partial_t - \frac{a}{\ell^2}\partial_\beta). \tag{4.117}$$

Identifying

$$\mathcal{P} = \gamma^3\mathcal{P}_0 = \frac{m\gamma^3}{8\pi\ell^2}, \quad \gamma = \Omega^{-1} = \frac{\sqrt{\Delta_P}}{P} = \frac{1}{\sqrt{1 - \frac{a^2}{\ell^2}e^{-2A}}}, \quad \beta = -\frac{a}{\ell^2}, \quad r = \ell, \tag{4.118}$$

we have found the black hole that is dual to the fluid flow with identical amounts of translational and rotational flow on the 2-hyperbola (3.80). As before, the region of the final conformal boundary (4.114) matches exactly with the region in which the fluid lives (3.79).

4.1.9 Case 9: $k < 0$

When

$$k < 0, \tag{4.119}$$

then

$$\Delta > 0, \quad \sqrt{\Delta} > |\epsilon|, \quad \alpha_+ > 0, \quad \alpha_- < 0, \tag{4.120}$$

giving

$$P(p) = \frac{1}{\ell^2}(p + \sqrt{\alpha_+})(p - \sqrt{\alpha_+})(p^2 - \ell^2\alpha_-), \tag{4.121}$$

which is always positive when for $|p| > \sqrt{\alpha_+}$.

4.1.9.1 Black hole dual to rotational flow on the 2-hyperbola (the hyperbolic Kerr–AdS black hole)

We can rescale by setting $\alpha_- = -\ell^2$ and defining $a^2 := \alpha_+$. Setting the following on the metric (4.1),

$$k = -a^2, \quad \epsilon = -1 + \frac{a^2}{\ell^2}, \quad \tau = t - a\phi, \quad q = r, \quad p = a\cosh\theta, \quad \sigma = -\frac{\phi}{a}, \tag{4.122}$$

obtaining

$$ds^2 = -\frac{\Delta_r}{\rho^2}(dt + a\sinh^2\theta d\phi)^2 + \rho^2\left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right) + \frac{\Delta_\theta \sinh^2\theta}{\rho^2}(adt - (r^2 + a^2)d\phi)^2, \tag{4.123}$$

where

$$\rho^2 = r^2 + a^2\cosh^2\theta, \quad \Delta_r = (r^2 + a^2)\left(-1 + \frac{r^2}{\ell^2}\right) - 2mr, \quad \Delta_\theta = 1 + \frac{a^2}{\ell^2}\cosh^2\theta. \tag{4.124}$$

This is the hyperbolic Kerr–AdS black hole [6]. The boundary would be

$$d\check{s}^2 = \frac{\Delta_r}{\rho^2}\left(-dt^2 + \frac{\ell^2}{\Delta_\theta}d\theta^2 + \sinh^2\theta\left(\ell^2 + a^2\right)d\phi^2 - 2a\sinh^2\theta\, dt\, d\phi\right). \tag{4.125}$$

After a conformal transformation,

$$d\hat{s}^2 = -dt^2 + \frac{\ell^2}{\Delta_\theta}d\theta^2 + \sinh^2\theta\left(\ell^2 + a^2\right)d\phi^2 - 2a\sinh^2\theta\, dt\, d\phi. \tag{4.126}$$

We use the following coordinates,

$$\cosh\Theta = \cosh\theta\sqrt{\frac{\Xi}{\Delta_\theta}}, \quad \Phi = \Xi\phi - \frac{at}{\ell^2}, \tag{4.127}$$

where

$$\Xi = 1 + \frac{a^2}{\ell^2}, \tag{4.128}$$

yielding

$$d\hat{s}^2 = \frac{\Delta_\theta}{\Xi}\left(-dt^2 + \ell^2(d\Theta^2 + \sinh^2\Theta\, d\Phi^2)\right), \tag{4.129}$$

which we then conformally transform with

$$\tilde{g}^{\mu\nu} = \Omega^2\hat{g}^{\mu\nu} = \frac{\Xi}{\Delta_\theta}\hat{g}^{\mu\nu}, \tag{4.130}$$

obtaining the 2-hyperbola

$$d\tilde{s}^2 = -dt^2 + \ell^2(d\Theta^2 + \sinh^2\Theta\, d\Phi^2). \tag{4.131}$$

Notice that

$$\begin{aligned}\lim_{\theta\to\infty}\cosh^2\Theta = \lim_{\theta\to\infty}\cosh^2\theta\frac{1+\frac{a^2}{\ell^2}}{1+\frac{a^2}{\ell^2}\cosh^2\theta} = \frac{\ell^2}{a^2}+1,\\ \lim_{\theta\to\infty}\sinh^2\Theta = \frac{\ell^2}{a^2},\end{aligned} \tag{4.132}$$

which means that the range of Θ is given by

$$0 \le \theta \quad \to \quad 0 \le \Theta < \frac{\ell}{|a|}. \tag{4.133}$$

The associated stress tensor to (4.126) would be

$$\hat{T}^{\mu\nu} = \frac{m}{8\pi\ell^2}(\hat{g}^{\mu\nu} + 3u^\mu u^\nu), \tag{4.134}$$

with $u = \partial_t$, using (4.28), which becomes the stress tensor associated with (4.131),

$$\tilde{T}^{\mu\nu} = \Omega^{-5}\hat{T}^{\mu\nu} = \frac{m\gamma^3}{8\pi\ell^2}(\tilde{g}^{\mu\nu} + 3\tilde{u}^\mu\tilde{u}^\nu), \tag{4.135}$$

with

$$\tilde{u} = \Omega^{-1} u = \gamma \left(\partial_t - \frac{a}{\ell^2} \partial_\Phi \right) . \tag{4.136}$$

We identify

$$\mathcal{P} = \gamma^3 \mathcal{P}_0 = \frac{m\gamma^3}{8\pi\ell^2}, \quad \omega = -\frac{a}{\ell^2}, \quad r = \ell, \quad \gamma = \Omega^{-1} = \sqrt{\frac{\Delta_\theta}{\Xi}} = \frac{1}{\sqrt{1 - \omega^2 r^2 \sinh^2 \Theta}}, \tag{4.137}$$

thus mapping this black hole to that of a rotating fluid in the hyperbolic plane (3.51). As always, the regions in which both cases are valid match perfectly, as seen in (3.50) and (4.133).

Chapter 5

A New Form of $P(p)$: $\tilde{P}(\tilde{p})$

At this stage, we have fully exhausted the different possibilities of $P(p)$ in its current form, and have constructed all the black holes found in [6]. However, two cases of fluid flow, namely that of translating fluid on the Euclidean plane and rotating fluid on the 2-sphere that completely covers a hemisphere, have not been mapped. The way to do so is to introduce a new form of $P(p)$, as detailed below.

If a rescaling and shift of p is conducted as follows,

$$p = l + a\tilde{p} \tag{5.1}$$

as motivated by Griffiths and Podolský [9], where l is the true NUT charge and a the rotation parameter, as seen in the spherical Kerr black hole with $p = a\cos\theta$. If $n = 0$ is still preserved, for the condition of conformally flat, we now have

$$\begin{aligned} P(\tilde{p}) &= k - \epsilon(l + a\tilde{p})^2 + \frac{(l + a\tilde{p})^4}{\ell^2} \\ &= k - \epsilon(l^2 + 2al\tilde{p} + a^2\tilde{p}^2) + \frac{1}{\ell^2}(l^4 + 4l^3a\tilde{p} + 6l^2a^2\tilde{p}^2 + 4la^3\tilde{p}^3 + a^4\tilde{p}^4) \\ &= k - \epsilon l^2 + \frac{l^4}{\ell^2} + \left(\frac{4l^3a}{\ell^2} - 2\epsilon al\right)\tilde{p} + \left(\frac{6l^2a^2}{\ell^2} - \epsilon a^2\right)\tilde{p}^2 + \frac{4la^3\tilde{p}^3}{\ell^2} + \frac{a^4\tilde{p}^4}{\ell^2} \end{aligned} \tag{5.2}$$

which (normally) does not change the physics in any way, as it is simply a mathematical rescaling and shift. However, as we will see below, this changes when certain limits are taken.

Now, we note that from the metric (4.1), the term regarding the p coordinate changes thus:

$$\frac{p^2+q^2}{P(p)}dp^2 = \frac{(l+a\tilde{p})^2+q^2}{P(\tilde{p})}a^2 d\tilde{p}^2 = \frac{(l+a\tilde{p})^2+q^2}{\tilde{P}(\tilde{p})}d\tilde{p}^2, \tag{5.3}$$

which in other words,

$$\tilde{P}(\tilde{p}) = \frac{P(p)}{a^2}. \tag{5.4}$$

Following Griffiths and Podolský [9], we introduce the appropriate coordinates in the Carter–Plebański metric (4.1) to use with $\tilde{P}(\tilde{p})$, given by

$$p = l + a\tilde{p}, \quad \tau = \tilde{\tau} - \frac{(l+a)^2}{a}\phi, \quad \sigma = -\frac{\phi}{a}, \tag{5.5}$$

changing the metric (4.1) to

$$\begin{aligned}
ds^2 &= \frac{p^2+q^2}{P(p)}dp^2 + \frac{P(p)}{p^2+q^2}(d\tau + q^2 d\sigma)^2 + \frac{p^2+q^2}{Q(q)}dq^2 - \frac{Q(q)}{p^2+q^2}(d\tau - p^2 d\sigma)^2 \\
&= \frac{(l+a\tilde{p})^2+q^2}{\tilde{P}(\tilde{p})}d\tilde{p}^2 + \frac{a^2\tilde{P}(\tilde{p})}{(l+a\tilde{p})^2+q^2}(d\tilde{\tau} - \frac{(l+a)^2}{a}d\phi - \frac{q^2}{a}d\phi)^2 \\
&\quad + \frac{(l+a\tilde{p})^2+q^2}{Q(q)}dq^2 - \frac{Q(q)}{(l+a\tilde{p})^2+q^2}(d\tilde{\tau} - \frac{(l+a)^2}{a}d\phi + \frac{(l+a\tilde{p})^2}{a}d\phi)^2 \\
&= \frac{(l+a\tilde{p})^2+q^2}{\tilde{P}(\tilde{p})}d\tilde{p}^2 + \frac{\tilde{P}(\tilde{p})}{(l+a\tilde{p})^2+q^2}(a d\tilde{\tau} - ((l+a)^2+q^2)d\phi)^2 \\
&\quad + \frac{(l+a\tilde{p})^2+q^2}{Q(q)}dq^2 - \frac{Q(q)}{(l+a\tilde{p})^2+q^2}(d\tilde{\tau} + (a(\tilde{p}-1) + 2l(\tilde{p}-1))d\phi)^2.
\end{aligned} \tag{5.6}$$

We note that these coordinates were introduced in obtaining the Kerr–AdS black hole, with the identification of $\tilde{p} = \cos\theta$, except with the added condition of $l = 0$.

5.1 Case 10: $k = a^2, \ \epsilon = 0, \ l = 0$, then limit $a = 0$.

If we set

$$k = a^2, \quad \epsilon = 0, \quad l = 0, \tag{5.7}$$

we get

$$P(\tilde{p}) = a^2 + \frac{a^4\tilde{p}^4}{\ell^2}. \tag{5.8}$$

Thus

$$\tilde{P}(\tilde{p}) = 1 + \frac{a^2\tilde{p}^4}{\ell^2}. \tag{5.9}$$

Set $a = 0$. We now have

$$\tilde{P}(\tilde{p}) = 1. \tag{5.10}$$

This is obviously a form that could not have been obtained under the previous classification.

We note that if $a = 0$ was set first, then the coordinate $\tilde{p}$ would simply disappear. The values of (5.7) were also chosen to remove any term in $\tilde{P}(\tilde{p})$ with a in the denominator, so as to prevent singularities when the limit $a = 0$ is imposed.

This is similar to the way the spherical Schwarzschild solution is obtained from the Carter–Plebański class, where we first obtain the Kerr solution, before setting $a = 0$.

5.1.1 Black hole dual to translational flow on the Euclidean plane (the planar Schwarzschild–AdS black hole)

From the rescaled Carter–Plebański metric (5.6), set

$$k = a^2, \quad \epsilon = 0, \quad \tilde{p} = x, \quad \phi = y, \quad q = r, \quad \tilde{\tau} = t, \quad l = 0, \tag{5.11}$$

then set $a = 0$, the metric is now

$$ds^2 = -\frac{\Delta_r}{r^2}dt^2 + \frac{r^2 dr^2}{\Delta_r} + r^2(dx^2 + dy^2), \tag{5.12}$$

where

$$\Delta_r = Q(q) = \frac{r^4}{\ell^2} - 2mr. \tag{5.13}$$

This is simply the planar Schwarzschild–AdS black hole [18]. We take the boundary at infinity,

$$d\check{s}^2 = \frac{r^2}{\ell^2}(-dt^2 + \ell^2(dx^2 + dy^2)), \tag{5.14}$$

becoming

$$d\hat{s}^2 = -dt^2 + \ell^2(dx^2 + dy^2). \tag{5.15}$$

Transform

$$X = \ell x. \quad Y = \ell y, \tag{5.16}$$

obtaining

$$d\hat{s}^2 = -dt^2 + dX^2 + dY^2. \tag{5.17}$$

The associated stress tensor to (5.17) is the standard

$$\hat{T}^{\mu\nu} = \frac{m}{8\pi\ell^2}(\hat{g}^{\mu\nu} + 3u^\mu u^\nu), \tag{5.18}$$

with $u = \partial_t$. This is dual to a stationary fluid on the Euclidean plane, with the argument of (4.28) still holding true, with the identification

$$\frac{m}{8\pi\ell^2} = \mathcal{P}_0. \tag{5.19}$$

Giving a Lorentz boost in the x-axis to the stationary fluid, we get

$$\tilde{u} = \gamma(\partial_t + \beta\partial_X), \tag{5.20}$$

with

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}. \tag{5.21}$$

The stress tensor is now

$$\tilde{T}^{\mu\nu} = \frac{m\gamma^3}{8\pi\ell^2}(\hat{g}^{\mu\nu} + 3\tilde{u}^\mu\tilde{u}^\nu), \tag{5.22}$$

where we used (2.61), identifying it with fluid with pure translational flow on the Euclidean plane (3.27). This is a well known result [19] of a black hole dual to fluid flow, as it is the most basic form of fluid flow.

5.2 Case 11: $k = \frac{l^4}{\ell^2} - 2al$, $\epsilon = \frac{2l^2}{\ell^2} - \frac{2a}{l}$, then limit $a = 0$.

Motivated by the earlier example, we now set

$$k = \frac{l^4}{\ell^2} - 2al, \quad \epsilon = \frac{2l^2}{\ell^2} - \frac{2a}{l}, \tag{5.23}$$

we get

$$P(\tilde{p}) = \frac{l^4}{\ell^2} - 2al - \left(\frac{2l^2}{\ell^2} - \frac{2a}{l}\right) l^2 + \frac{l^4}{\ell^2} + \left(\frac{4l^3 a}{\ell^2} - 2\left(\frac{2l^2}{\ell^2} - \frac{2a}{l}\right) al\right) \tilde{p}$$
$$+ \left(\frac{6l^2 a^2}{\ell^2} - \left(\frac{2l^2}{\ell^2} - \frac{2a}{l}\right) a^2\right) \tilde{p}^2 + \frac{4la^3\tilde{p}^3}{\ell^2} + \frac{a^4\tilde{p}^4}{\ell^2}$$
$$= 4a^2\tilde{p} + \left(\frac{4l^2a^2}{\ell^2} - \frac{2a^3}{l}\right)\tilde{p}^2 + \frac{4la^3\tilde{p}^3}{\ell^2} + \frac{a^4\tilde{p}^4}{\ell^2}. \tag{5.24}$$

The quartic $Q(q)$ becomes

$$Q(q) = \frac{l^4}{\ell^2} - 2al - 2mq + \left(\frac{2l^2}{\ell^2} - \frac{2a}{l}\right) q^2 + \frac{q^4}{\ell^2}$$
$$= \frac{(q^2+l^2)^2}{\ell^2} - 2mq - 2al - \frac{2a}{l}q^2. \tag{5.25}$$

Now, if we use the values of (5.23),

$$\tilde{P}(\tilde{p}) = \frac{P(\tilde{p})}{a^2} = 4\tilde{p} + \left(\frac{4l^2}{\ell^2} - \frac{2a}{l}\right)\tilde{p}^2 + \frac{4la\tilde{p}^3}{\ell^2} + \frac{a^2\tilde{p}^4}{\ell^2}. \tag{5.26}$$

If we now take the limit $a = 0$, we have

$$\tilde{P}(\tilde{p}) = 4\tilde{p} + \frac{4l^2}{\ell^2}\tilde{p}^2. \tag{5.27}$$

As before, this is missing in the original classification, and only appears with the specific values set with (5.23) and the limit $a = 0$, and only in that order. The values of (5.7) were also chosen to remove any term in $\tilde{P}(\tilde{p})$ with a in the denominator, but also to retain a non-zero l.

5.2.1 Black hole dual to rotational fluid flow on a hemisphere of the 2-sphere

Setting,

$$\tilde{p} = \psi, \quad q = r, \quad \phi = \zeta, \quad \tilde{\tau} = t + 2l\zeta, \quad k = \frac{l^4}{\ell^2} - 2al, \quad \epsilon = \frac{2l^2}{\ell^2} - \frac{2a}{l}, \tag{5.28}$$

then $a = 0$, as in the earlier discussion, (5.6) is now

$$ds^2 = -\frac{\Delta_r}{\rho^2}(dt + 2l\psi d\zeta)^2 + \frac{\rho^2 dr^2}{\Delta_r} + \frac{\rho^2 d\psi^2}{\Delta_\psi} + \rho^2\Delta_\psi d\zeta^2, \tag{5.29}$$

where, using (5.25) and (5.4),

$$\Delta_r = Q(q) = \frac{\rho^4}{\ell^2} - 2mr, \quad \Delta_\psi = \tilde{P}(\tilde{p}) = 4\psi + \frac{4l^2}{\ell^2}\psi^2, \quad \rho^2 = r^2 + l^2. \tag{5.30}$$

To verify that this black hole has the correct metric signature, we note that

$$\begin{aligned} g_{\zeta\zeta} &= \rho^2 \Delta_\psi - \frac{\Delta_r}{\rho^2} 4l^2\psi^2 \\ &= \rho^2 \left(4\psi + \frac{4l^2}{\ell^2}\psi^2\right) - \left(\frac{\rho^4}{\ell^2} - 2mr\right) \frac{4l^2\psi^2}{\rho^2} \\ &= 4\psi\rho^2 \left(1 + \frac{2mrl^2\psi}{\rho^4}\right), \end{aligned} \tag{5.31}$$

which means that the correct signature is displayed when

$$\frac{\rho^4}{\ell^2} > 2mr, \qquad \psi > 0. \tag{5.32}$$

Extracting the boundary metric,

$$\begin{aligned} d\check{s}^2 &= -\frac{\Delta_r}{\rho^2}(dt + 2l\psi d\zeta)^2 + \frac{\rho^2 d\psi^2}{\Delta_\psi} + \rho^2 \Delta_\psi d\zeta^2 \\ &= -\frac{\rho^2}{\ell^2}(dt + 2l\psi d\zeta)^2 + \frac{\rho^2 d\psi^2}{\Delta_\psi} + \rho^2 \Delta_\psi d\zeta^2 \\ &= \frac{\rho^2}{\ell^2}\left(-dt^2 - 4l\psi dt d\zeta - 4l^2\psi^2 d\zeta^2 + \frac{\ell^2}{\Delta_\psi} d\psi^2 + \ell^2 \Delta_\psi d\zeta^2\right) \\ &= \frac{\rho^2}{\ell^2}\left(-dt^2 - 4l\psi dt d\zeta + \frac{\ell^2}{\Delta_\psi} d\psi^2 + 4\ell^2 \psi d\zeta^2\right), \end{aligned} \tag{5.33}$$

performing the conformal transformation,

$$d\hat{s}^2 = -dt^2 - 4l\psi dt d\zeta + \frac{\ell^2}{\Delta_\psi} d\psi^2 + 4\ell^2\psi d\zeta^2. \tag{5.34}$$

We now provide new coordinates,

$$\psi = \frac{\ell^2}{l^2}\tan^2\Psi, \quad Z = 2\zeta - \frac{l}{\ell^2}t, \tag{5.35}$$

which gives

$$d\psi = 2\frac{\ell^2}{l^2}\tan\Psi \sec^2\Psi d\Psi, \quad d\zeta = \frac{1}{2}dZ + \frac{l}{2\ell^2}dt, \tag{5.36}$$

making the boundary

$$\begin{aligned} d\hat{s}^2 &= -dt^2 - 4l\psi dt\left(\frac{1}{2}dZ + \frac{l}{2\ell^2}dt\right) + \frac{\ell^2}{\Delta_\psi}d\psi^2 + 4\ell^2\psi\left(\frac{1}{2}dZ + \frac{l}{2\ell^2}dt\right)^2 \\ &= -dt^2 - 2l\psi dtdZ - \frac{2l^2}{\ell^2}\psi dt^2 + \frac{\ell^2}{\Delta_\psi}d\psi^2 + \ell^2\psi dZ^2 + 2l\psi dtdZ + \frac{l^2}{\ell^2}\psi dt^2 \\ &= -\left(1 + \frac{l^2}{\ell^2}\psi\right)dt^2 + \frac{\ell^2}{\Delta_\psi}d\psi^2 + \ell^2\psi dZ^2 \\ &= \left(1 + \frac{l^2}{\ell^2}\psi\right)\left(-dt^2 + \frac{\ell^2}{4\psi(1 + \frac{l^2}{\ell^2}\psi)^2}d\psi^2 + \frac{\ell^2}{(1 + \frac{l^2}{\ell^2}\psi)}\psi dZ^2\right) \\ &= \frac{1}{\cos^2\Psi}\left(-dt^2 + \frac{\ell^4}{l^2}(d\Psi^2 + \sin^2\Psi dZ^2)\right), \end{aligned} \tag{5.37}$$

where we took into account that

$$\frac{l^2}{\ell^2}\psi = \tan^2\Psi = \frac{\sin^2\Psi}{\cos^2\Psi} = \frac{1 - \cos^2\Psi}{\cos^2\Psi} = \frac{\sin^2\Psi}{1 - \sin^2\Psi} \tag{5.38}$$

and thus

$$\begin{aligned} \cos^2\Psi &= \frac{1}{1 + \frac{l^2}{\ell^2}\psi}, \quad \sin^2\Psi = \frac{\frac{l^2}{\ell^2}\psi}{1 + \frac{l^2}{\ell^2}\psi}, \\ \frac{l^2\cos^4\Psi}{4\tan^2\Psi} &= \frac{\ell^2}{4\psi(1 + \frac{l^2}{\ell^2}\psi)^2}, \quad \frac{\ell^4}{l^2}\sin^2\Psi = \frac{\ell^2}{(1 + \frac{l^2}{\ell^2}\psi)}\psi. \end{aligned} \tag{5.39}$$

This metric is simply the 2-sphere up to a conformal transformation. We see that Ψ takes the range of $0 < \Psi < \frac{\pi}{2}$, due to (5.35).

The appropriate conformal transformation is thus

$$\tilde{g}^{\mu\nu} = \Omega^2\hat{g}^{\mu\nu} = \cos^2\Psi\hat{g}^{\mu\nu}. \tag{5.40}$$

This would cause the initial stationary fluid (again, note the argument of (4.28)) to transform into

$$\begin{aligned} u^\mu &= \partial_t \\ &= \begin{bmatrix} \frac{\partial t}{\partial t} & \frac{\partial t}{\partial \psi} & \frac{\partial t}{\partial \zeta} \\ \frac{\partial \Psi}{\partial t} & \frac{\partial \Psi}{\partial \psi} & \frac{\partial \Psi}{\partial \zeta} \\ \frac{\partial Z}{\partial t} & \frac{\partial Z}{\partial \psi} & \frac{\partial Z}{\partial \zeta} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \\ -\frac{l}{\ell^2} \end{bmatrix} = \partial_t - \frac{l}{\ell^2}\partial_Z, \end{aligned} \tag{5.41}$$

as

$$\frac{\partial t}{\partial t} = 1, \quad \frac{\partial \Psi}{\partial t} = 0, \quad \frac{\partial Z}{\partial t} = -\frac{l}{\ell^2}. \tag{5.42}$$

After the conformal transformation, we then have

$$\tilde{u}^\mu = \frac{1}{\cos\Psi}\left(\partial_t - \frac{l}{\ell^2}\partial_Z\right), \tag{5.43}$$

as $\tilde{u}^\mu = \Omega^{-1}u^\mu$.

Knowing that the velocity of a conformal fluid rotating on the 2-sphere we have

$$u^\mu = \gamma(\partial_t + \omega\partial_\phi) = \frac{1}{\sqrt{1 - \omega^2 r^2 \sin^2\theta}}(\partial_t + \omega\partial_\phi), \tag{5.44}$$

thus allowing us to identify

$$r^2 = \frac{\ell^4}{l^2}, \quad \gamma = \Omega^{-1} = \frac{1}{\cos\Psi}, \quad \omega = -\frac{l}{\ell^2}. \tag{5.45}$$

The stress tensor associated with (5.34) is

$$\hat{T}^{\mu\nu} = \frac{m}{8\pi\ell^2}(\hat{g}^{\mu\nu} + 3u^\mu u^\nu), \tag{5.46}$$

with the stress tensor transforming to

$$\tilde{T}^{\mu\nu} = \frac{m\gamma^3}{8\pi\ell^2}(\tilde{g}^{\mu\nu} + 3\tilde{u}^\mu\tilde{u}^\nu), \tag{5.47}$$

where we identify again

$$\frac{m\gamma^3}{8\pi\ell^2} = \mathcal{P} = \gamma^3\mathcal{P}_0. \tag{5.48}$$

This covers the case of $|\omega| = \frac{1}{r}$ that is missing in [6] on the 2-sphere (3.31), and is a new result.

This corresponds to a fluid that exist only on one hemisphere, with the fluid approaching the speed to light as it gets closer to the hemisphere (3.32). This corresponds perfectly with the range the boundary metric is valid after the coordinate and conformal transformation.

One might argue that this case is a limit of the black hole dual to the polar region on the 2-sphere (4.36), but we note that the constrain of $a^2 > \ell^2$ is a strict inequality for (4.36) to retain its signature.

This black hole is a new solution that requires further study. It contains a non-zero NUT charge l, but the parameter that closely associated with it is set to $n = 0$. We also have l being related to the angular velocity of the fluid ω, a role that is usually related by a. However, in this special case, the angular velocity is fixed with respect to the geometry of the manifold, which in this case is also related to l, which in other cases were only related to the cosmological constant ℓ.

5.3 Remarks on $\tilde{P}(\tilde{p})$

It is clear that for this $\tilde{P}(\tilde{p})$ to be more than just a trivial shift and rescaling of $P(p)$, there needs to be a limit of $a = 0$. If the NUT charge $l = 0$, we note that $\tilde{P}(\tilde{p}) = 1 - \tilde{p}^2$ and $\tilde{P}(\tilde{p}) = -1 + \tilde{p}^2$ corresponds to the spherical and hyperbolic Schwarzschild–AdS black holes respectively, and can be obtained from their Kerr versions by setting $a = 0$.

The limit of $a = 0$ means that the $p^4 = a^4\tilde{p}^4$ term in the original $P(p)$ will vanish unless we impose $\ell^2 \propto a^2$, which is undesirable, as it means an infinitely large (negative) cosmological constant.

The relation between the true NUT charge l and the parameter n was clarified in [9] for standard compact black holes. However the relationship is still unclear with respect to extended black holes. The standard NUT charge leads to closed timelike curves [17], and is thus generally an unwanted parameter. In case 11, we have constructed a black hole with a vanishing n but a non-vanishing l, which does not occur in the standard NUT solution [9]. This deserves further study.

Chapter 6

Conclusion

We now have a complete classification of the black holes (in 3+1 dimensions) dual to fluid flows on ultrastatic manifolds (in 2+1 dimensions) with constant curvature on the spatial sections. This was done by first constructing a proof that equates equilibrium fluid flows to Killing fields. Next, this proof was used to classifying the various fluid flows. Finally, the black holes that are dual to the flows were constructed from the Carter–Plebański class of solutions.

The previous classification [6] lacked the black holes dual to two fluid flows, namely, that of purely translating fluid on the Euclidean plane (which was mentioned but not elaborated on) and rotating fluid on the 2-sphere that completely covers a hemisphere. While it is true that the purely translating fluid on the Euclidean plane is well studied in the context of of gravity/gauge duality, it could not be identified in the context of the classification via the quartic $P(p)$ of the Carter–Plebański Class.

In order to identify it, a scaling limit to obtain $\tilde{P}(\tilde{p})$ was required. This scaling limit was then also used to identity the black hole dual to rotating fluid on the 2-sphere that completely covers a hemisphere, which is a new result.

This thesis has thus provided a new entry into the "dictionary" of gravity/fluid, as well as providing an insight of how the quartic $P(p)$ in the Carter–Plebański Class affects black holes, in the context of AdS, where black holes with different geometry from that of the standard spherical ones exist.

It would be interesting to further study the Carter–Plebański Class via the rescaled $\tilde{P}(\tilde{p})$. Further classification of all the possible cases of $\tilde{P}(\tilde{p})$ could be attempted. It might also be interesting to relax the condition $n = 0$, allowing us to map to fluids on non-conformally flat manifolds. It would be interesting to see what fluid the Taub–NUT metric is dual to. Another possible extension would be to extend to higher dimensions, with the 4+1 dimensional black holes dual to 3+1 dimensional fluid flows the one of most interest.

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