Discrete Symmetries in Particle Physics

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Abstract

We wish to generate the CP-violating effects in neutrino oscillations. This is done by firstly constructing modified Lagrangians that break CP symmetry by imposing specific constraints. This violation in CP stems from the Lorentz-violating field that enters through the additional factor F that makes our Lagrangian different from that of the typical. This leads to a change in the energy dispersion relation and in turn affects the neutrino oscillation probability. With the aid of experimental results; we can give an upper bound to the background field. This limit on the background field is useful when analysing data that deviates from the conventional. These data can be fitted with our results to see if it agrees with our calculations. If it does, then CP violation in neutrino oscillations could be due to what we have proposed.

Chapter 1

Introduction

In Particle Physics, CP violation refers to the non-observance of CP-symmetry, which is essentially the combination of the charge conjugation (C) symmetry and parity (P) symmetry. It was first detected in 1964 by James Cronin and Val Fitch while they were studying the decay of neutral kaons. Since then, it has sparked the interest of particle physicists as it can hold key to solving fundamental problems that have been plaguing us, such as the fundamental issue of matter-antimatter asymmetry. CP violation arises naturally in the three-generation Standard Model but as mentioned in [1], it is not likely that it presents the description of CP violation in nature entirely. CP violation is observed in the decay of neutral kaons and B mesons, possibly in neutrinos as well. Physicists have an inkling where to find them: neutrino oscillations.

The phenomenology of neutrino oscillations is one in which neutrinos of a particular flavour are observed to morph into another flavour after propagating a certain distance. CP violation enters neutrino oscillation through the third neutrino mixing angle, θ_{13} and manifests itself as the CP violating phase, δ. Today, there are many on-going experiments that try to determine this parameter, such as the T2K (Tokai to Kamioka) experiment in Japan. However, it is unclear as to where CP violation originated in neutrino oscillations.

CP violation arises naturally in the three-generation Standard Model but as mentioned in [1], it is not likely that it presents the description of CP violation in nature entirely. It is apparent that new physics exist beyond the Standard Model and such extensions often have 'additional sources of CP-violating effects'. Lorentz violation, as suggested by some quantum gravity model, belongs to such a framework which includes operators that break or preserve CPT. Thus, the breaking of Lorentz symmetry may imply the CPT violation and can hint at the breaking of CP symmetry. [2, 3, 4] Hence, our objective is to use the Standard Model Extension to generate CP-violating effects in neutrino oscillations.

We wish to construct CP-violating Lagrangians of the form

$$\mathcal{L} = \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi + F \tag{1.1}$$

where F contains the Lorentz-violating background field. The Lagrangians are constructed by imposing constraints similar to the typical Dirac Lagrangian; but with the additional condition that they must break CP symmetry. These modified Lagrangians are interpreted as new physics that are yet unbeknownst to us. They will lead to a change in the energy dispersion relation as well, thereby affecting the neutrino oscillation probabilities. With values of the various parameters that were already measured from experiments, we can determine bounds of the background field, which will be useful in the future when we compare with new experimental data.

This thesis is outlined as follows. In Chapter 2, we describe the axiomatic approach to construct the desired modified Lagrangians and look at the specific constraints that were applied. In Chapter 3, we give the description of discrete symmetries and how each modified Lagrangian transforms under the different symmetry. Moving on to Chapter 4, we will be deriving the plane-wave solutions of the Dirac Lagrangian and thereafter derive the modified energy dispersion relations. In Chapter 5, we delve into neutrino oscillations where we first look at the conventional theory, and apply the modified dispersion relations to see the changes in oscillation probabilities.

Thereafter, we determine the bounds of the background fields and discuss the results obtained. Finally, we conclude with a summary in Chapter 6.

Chapter 2

Axiomatic Approach to the Construction of the Modified Lagrangian

In this chapter, we will discuss how the modified Dirac Lagrangian that violates CP symmetry is formulated by imposing particular constraints on the additional term. These constraints are typical properties that the conventional Dirac Lagrangian [$\mathcal{L} = \frac{i}{2}\bar{\psi}\big(i\gamma^{\mu}\partial_{\mu} - i\gamma^{\mu}\overleftarrow{\partial_{\mu}}\big)\psi + \bar{\psi}m\psi$] possesses. We will then discuss how Lorentz violation is introduced into the Lagrangian. At the end of the chapter, we will look at six specific examples of the CP-violating Lagrangians, which will be of the following form:

$$\mathcal{L} = \bar{\psi}\big(i\gamma^{\mu}\partial_{\mu} - m\big)\psi + F \tag{2.1}$$

where F is the additional term that we add in.

2.1 Constraints

Now we examine the different constraints that will be imposed on F. These properties are those that are required of the usual Dirac Lagrangian, except for one that we choose to violate; that of CP symmetry (which will be further discussed in Chapter 3). We thus require our Lagrangians to have the following properties:

- *Hermiticity*

  Just as in Quantum Mechanics; which requires the Hamiltonian to be Hermitian, F should be Hermitian as well. This is to ensure that the eigenvalues and hence the eigen-energies are real. This indicates that the following equation should hold,

$$F^{+} = F. \tag{2.2}$$

- *Locality*

  Physics that is described by a wavefunction will be accurately captured by a local evolution equation. This will continue to be the case in our situation. In this way, F will only depend on the wavefunction, its adjoint and their derivatives all evaluated at a single point.

- *Universality*

  It is a phenomenon in which the physics remains unchanged even when the wavefunction undergoes rescaling. With this scale invariance property, F should be of the same form whether it describes a single particle or a system of particles.

There is another constraint arising from discrete symmetries that requires our modified Lagrangians to be CP-violating but this will be further discussed in Section 3 as mentioned.

2.2 Lorentz Violation

As aforementioned, our objective is to generate CP violation in neutrino oscillations. One possible explanation for it will be due to the breakdown of Lorentz symmetry. There are certain quantum gravity models that suggest Lorentz violation and they belong to a framework that extends beyond the Standard Model, one that includes operators that violates CPT symmetry. [2] For example, certain string theories could cause the spontaneous breaking of CPT symmetry. [3] So the breaking of Lorentz symmetry may imply that CPT symmetry is broken too, and can then indirectly hint at the violation of CP symmetry. [4]

In our case, Lorentz violation enters through the constant background field in the form of $A_\mu = (A, 0,0,0)$. This background field preserves the observer Lorentz symmetry but the particle Lorentz symmetry is broken. By definition, observer Lorentz transformations are enforced by coordinate changes whereas particle Lorentz transformations relate the properties of two particles with different spin orientation or momentum within a specific oriented inertial frame and it includes boosts on particles or localized fields but *not* background fields. This is illustrated in Figure 1 below.

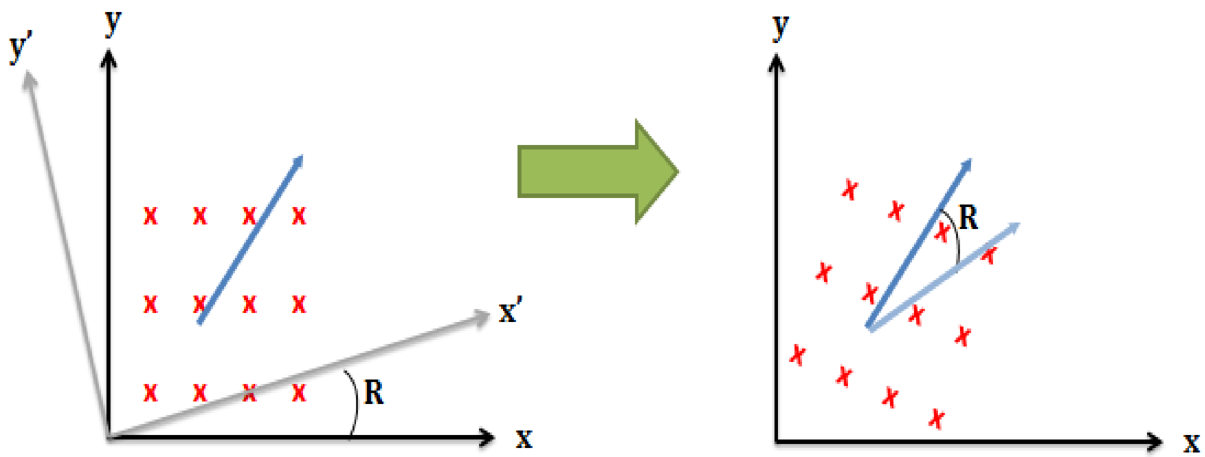

Figure 1(a): Observer Lorentz Transformation whereby coordinate change is involved, thus Lorentz symmetry still holds.

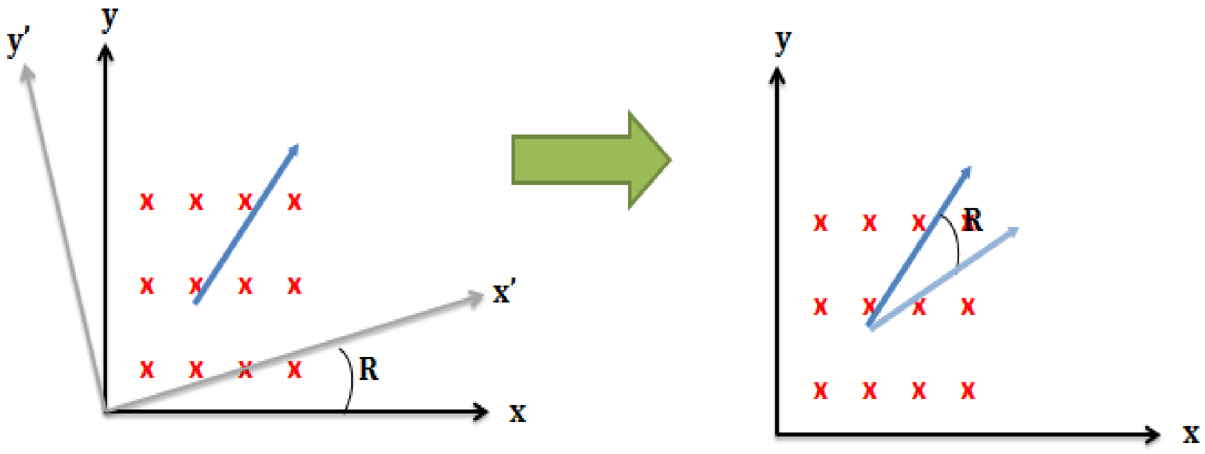

Figure 1(b): Particle Lorentz Transformation which involves boosts on particles but *not* background field, thus Lorentz symmetry is broken.

Thus the background field preserves the observer Lorentz symmetry but violates particle Lorentz symmetry.

2.3 Explicit Examples

In this section, we will look at six specific examples of modified Lorentz-violating Dirac Lagrangians that specify the three constraints mentioned in Section 2.1. As a reminder, the Lorentz violation enters through the background field, A_μ, which is of the form (A,0,0,0). It does not mean that A_μ is manifestly covariant, as per popular sentiment; it is simply a formalism whereby A_μ is a scalar with a Lorentz index. To reinforce the fact that A_μ is of this particular form, we will verify it by applying discrete symmetries to the modified Lagrangians (in Chapter 3).

Thus our altered Lagrangians are shown as below

$$\mathcal{L}_1 = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \; + A_\mu \bar{\psi}\gamma^\mu \psi \tag{2.3}$$

in which $F_1 = A_\mu \bar{\psi}\gamma^\mu \psi$.

$$\mathcal{L}_2 = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + B_\mu \bar{\psi}\gamma^\mu \gamma^5 \psi \tag{2.4}$$

in which $F_2 = B_\mu \bar{\psi}\gamma^\mu \gamma^5 \psi$.

$$\mathcal{L}_3 = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + iC_\mu(\partial_\mu \bar{\psi})\psi - iC_\mu \bar{\psi}(\partial_\mu \psi) \tag{2.5}$$

in which $F_3 = iC_\mu(\partial_\mu \bar{\psi})\psi - iC_\mu \bar{\psi}(\partial_\mu \psi)$.

$$\mathcal{L}_4 = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + iD_\mu \gamma^\mu(\partial_\mu \bar{\psi})\psi - iD_\mu \gamma^\mu \bar{\psi}(\partial_\mu \psi) \tag{2.6}$$

in which $F_4 = iD_\mu \gamma^\mu(\partial_\mu \bar{\psi})\psi - iD_\mu \gamma^\mu \bar{\psi}(\partial_\mu \psi)$.

$$\mathcal{L}_5 = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \; + iG_\mu \gamma^5(\partial_\mu \bar{\psi})\psi - iG_\mu \gamma^5 \bar{\psi}(\partial_\mu \psi) \tag{2.7}$$

in which $F_5 = iG_\mu \gamma^5(\partial_\mu \bar{\psi})\psi - iG_\mu \gamma^5 \bar{\psi}(\partial_\mu \psi)$.

$$\mathcal{L}_6 = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \; + A_{\mu\nu} \bar{\psi}\sigma^{\mu\nu} \psi \tag{2.8}$$

in which $F_6 = A_{\mu\nu} \bar{\psi}\sigma^{\mu\nu} \psi$.

Chapter 3

Discrete Symmetries

The Standard Model is indeed telling of CP violation; however it is contrary that it provides the description of CP violation in its entirety. [6] Henceforth, we look beyond the Standard Model to provide reason for CP violation in neutrino oscillation. In this section, we first look at the individual discrete symmetries and how the modified Lagrangians transform under the different symmetries. Thereafter we will impose the condition of CP violation in the six lagrangians. This additional prerequisite will help us obtain the final, specific form of the modified Lagrangians.

The parity transformation, charge conjugation and time reversal operator is given below respectively.

$$\hat{P} = \eta_P \gamma^0 \tag{3.1}$$

$$\hat{C} = i\eta_C \gamma^2 \tag{3.2}$$

$$\hat{T} = i\eta_T \gamma^1 \gamma^3 \tag{3.3}$$

where η_P, η_C and η_T are unobservable arbitrary phases. [7]

3.1 Transformation of the Modified Lagrangians

In this section, we will be looking at how $F_3 = iC_\mu\left(\partial_\mu \bar{\psi}\right)\psi - iC_\mu \bar{\psi}(\partial_\mu \psi)$ transforms under the different symmetries only. For the rest of the Lagrangians, the calculations can be found in **Appendix A**.

3.1.1 Parity Transformation (P) of F_3

First, we look at how the individual components transform under parity. For the Dirac spinor and its adjoint, they transform as follows:

$$\psi' \overset{P}{\rightarrow} \gamma^0\psi \tag{3.4}$$

$$\bar{\psi}' \overset{P}{\rightarrow} \psi'^{+}\gamma^0 = \psi^{+}\gamma^0\gamma^0 = \bar{\psi}\gamma^0 \tag{3.5}$$

where the prime (i.e. ψ') denotes the transformed spinors.

The spatial part of the derivative transforms as well, and so the derivative becomes

$$\partial_\mu' = (\partial_0, -\partial_i). \tag{3.6}$$

Thus, the transformation of F_3 under parity is

$$\begin{aligned} F_{3P} &= iC_\mu\big(\partial_\mu'\bar{\psi}'\big)\psi' - iC_\mu\bar{\psi}'(\partial_\mu{}'\psi') \\ &= iC_\mu\big(\partial_\mu{}'\bar{\psi}\gamma^0\big)\gamma^0\psi - iC_\mu\bar{\psi}\gamma^0(\partial_\mu{}'\gamma^0\psi). \end{aligned}$$

Since parity affects the temporal and spatial part of the derivative differently, there will be two cases:

When $\mu = 0$*:*

$$\begin{aligned} F_{3P} &= iC_0(\partial_0\bar{\psi}\gamma^0)\gamma^0\psi - iC_0\bar{\psi}\gamma^0(\partial_0\gamma^0\psi) \\ &= iC_0(\partial_0\bar{\psi})\psi - iC_0\bar{\psi}\,(\partial_0\psi). \end{aligned} \tag{3.7}$$

When $\mu = i$*:*

$$\begin{aligned} F_{3P} &= -iC_i(\partial_i\bar{\psi}\gamma^0)\gamma^0\psi + iC_i\bar{\psi}\gamma^0(\partial_i\gamma^0\psi) \\ &= -iC_i(\partial_i\bar{\psi})\psi + iC_i\bar{\psi}\,(\partial_i\psi). \end{aligned} \tag{3.8}$$

We observe that P is even for $\mu = 0$, since the F_{3P} remains unchanged. Whereas for $\mu = i$, P is odd since it differs from the original with an additional negative sign.

3.1.2 Charge Conjugation (C) of F_3

As what we have done for P, we find out how the individual component changes under C. For the Dirac spinor and its adjoint, they transform as such,

$$\psi' \xrightarrow{C} i\gamma^2\psi^+ \tag{3.9}$$

$$\bar{\psi}' \xrightarrow{C} \psi'^+\gamma_0 = (i\gamma^2\psi^+)^+\gamma^0 = -i\gamma^2\psi\gamma^0. \tag{3.10}$$

The derivative in this case remains unchanged as it is not affected by charge conjugation.

Thus, the transformation of F_3 under charge conjugation is

$$\begin{aligned} F_{3C} &= -iC_\mu\big(\partial'_\mu\bar{\psi}\big)\psi' + iC_\mu\bar{\psi}'(\partial_\mu{}'\psi') \\ &= -iC_\mu\big(\partial_\mu\bar{\psi}\gamma^0\gamma^2\big)\gamma^0\gamma^2\psi + iC_\mu(\bar{\psi}\gamma^0\gamma^2)(\partial_\mu\gamma^0\gamma^2\psi) \\ &= -iC_\mu\big(\partial_\mu\bar{\psi}\big)\psi + iC_\mu\bar{\psi}(\partial_\mu\psi). \end{aligned} \tag{3.11}$$

It is observed that C is odd for F_3 since it is not the same as before charge conjugation was applied.

3.1.3 Time Reversal (T) of F_3

Again, we look at how the individual components transform under parity. For the Dirac spinor and its adjoint, they transform as follows:

$$\psi' \xrightarrow{T} i\gamma^1\gamma^3\psi \tag{3.12}$$

$$\bar{\psi}' \xrightarrow{T} = (i\gamma^1\gamma^3\psi)^+\gamma^0 = -i\gamma^1\gamma^3\bar{\psi}. \tag{3.13}$$

The temporal part of the derivative transforms under time reversal as well, and so the derivative becomes

$$\partial'_\mu = (-\partial_0, \partial_i). \tag{3.14}$$

Thus, the transformation of F_3 under time reversal is

$$\begin{aligned} F_{3T} &= iC_\mu\left(\partial'_\mu\bar{\psi}'\right)\psi' - iC_\mu\bar{\psi}'(\partial_\mu{}'\psi') \\ &= iC_\mu\left(\partial_\mu{}'\bar{\psi}i\gamma^1\gamma^3\right)i\gamma^1\gamma^3\psi - iC_\mu\bar{\psi}i\gamma^1\gamma^3(\partial_\mu{}'i\gamma^1\gamma^3\psi). \end{aligned}$$

Since time reversal affects the temporal and spatial part of the derivative differently, there will be two unique cases, just like the case for parity transformation.

When $\mu = 0$*:*

$$\begin{aligned} F_{3T} &= -iC_0(\partial_0\bar{\psi}i\gamma^1\gamma^3)i\gamma^1\gamma^3\psi + iC_0\bar{\psi}i\gamma^1\gamma^3(\partial_0 i\gamma^1\gamma^3\psi) \\ &= -iC_0(\partial_0\bar{\psi})\psi + iC_0\bar{\psi}\,(\partial_0\psi). \end{aligned} \tag{3.15}$$

When $\mu = i$*:*

$$\begin{aligned} F_{3T} &= iC_i(\partial_i\bar{\psi}i\gamma^1\gamma^3)i\gamma^1\gamma^3\psi - iC_i\bar{\psi}i\gamma^1\gamma^3(\partial_i i\gamma^1\gamma^3\psi) \\ &= iC_i(\partial_i\bar{\psi})\psi - iC_i\bar{\psi}\,(\partial_i\psi). \end{aligned} \tag{3.16}$$

We observe that T is odd for $\mu = 0$, since the F_{3T} differs from the original with an additional negative sign.. Whereas for $\mu = i$, T is even since it remains unchanged.

3.2 CP Violation

In order to achieve CP violation, i.e. CP-odd, there can be two cases: the first case in which *C is odd* while *P is even*; and the second case in which *C is even* whereas *P is odd*. For the case of F_3, P is even when $\mu = 0$ and odd when $\mu = i$; and C is odd for both cases. So in order to get CP-odd, it will only happen when $\mu = 0$. Thus the final expression of F_3 is

$$F_3 = iC_0(\partial_0\bar{\psi})\psi - iC_0\bar{\psi}(\partial_0\psi). \tag{3.17}$$

Table 1 below summarizes the results for the six modified lagrangians. Detailed derivations of how they are obtained can be found in **Appendix A**.

F_n	μ	P	C	T	CP	CPT	Final Form of Modified Lagrangian
F_1	0	+	-	+	-	-	$F_1 = A_0\bar{\psi}\gamma^0\psi$
	i	-	-	-	+	-	
F_2	0	-	+	+	-	-	$F_2 = B_0\bar{\psi}\gamma^0\gamma^5\psi$
	i	+	+	-	+	-	
F_3	0	+	-	-	-	+	$F_3 = iC_0(\partial_0\bar{\psi})\psi - iC_0\bar{\psi}(\partial_0\psi)$
	i	-	-	+	+	+	
F_4	0	+	-	-	-	+	$F_4 = iD_0\gamma^0(\partial_0\bar{\psi})\psi - iD_0\gamma^0\bar{\psi}(\partial_0\psi)$
	i	-	-	+	+	+	
F_5	0	+	-	-	-	+	$F_5 = iG_0\gamma^5(\partial_0\bar{\psi})\psi - iG_0\gamma^5\bar{\psi}(\partial_0\psi)$
	i	-	-	+	+	+	

Table 1: Summary of Modified Lagrangians.

As we can see from Table 1, F_1 and F_2 breaks CPT symmetry and this implies directly that Lorentz symmetry is broken. Whereas for F_3, F_4 and F_5, CPT symmetry is still preserved, we are unable to make a conclusive statement whether Lorentz is violated.

The reason is because CPT belongs to a larger symmetry group that includes Lorentz violation. We can have a Lorentz violating system that preserves CPT symmetry but not a CPT violating system that preserves Lorentz symmetry. [7]

One will also realise that F_6 is missing from the table. The reason is that it does not comply with the condition of CP violation and hence it is eliminated. We will now look at the mathematics behind that leads us to this conclusion.

We first find out how F_6 transforms under parity.

$$\begin{aligned} F_{6P} &= A_{\mu\nu}\bar{\psi}'\sigma^{\mu\nu}\psi' \\ &= \tfrac{i}{2}A_{\mu\nu}\bar{\psi}'(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})\psi' \\ &= \tfrac{i}{2}A_{\mu\nu}\bar{\psi}\gamma^{0}(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})\gamma^{0}\psi. \end{aligned}$$

Since there are two dummy variables (μ and ν), where they can be 0 or *i*, there will be a total of four scenarios.

Two of the same cases where μ= ν and can *both* be either 0 or *i*:

$$F_{6P} = \frac{i}{2}A_{\mu\mu}\bar{\psi}\gamma^{0}(\gamma^{\mu}\gamma^{\mu} - \gamma^{\mu}\gamma^{\mu})\gamma^{0}\psi = 0 \tag{3.18}$$

leading to a trivial solution.

When $\mu = 0$ and $\nu = i$:

$$\begin{aligned} F_{6P} &= \frac{i}{2}A_{0i}\bar{\psi}\gamma^{0}(\gamma^{0}\gamma^{i} - \gamma^{i}\gamma^{0})\gamma^{0}\psi \\ &= \tfrac{i}{2}A_{0i}(\bar{\psi}\gamma^{i}\gamma^{0}\psi - \bar{\psi}\gamma^{0}\gamma^{i}\psi) \\ &= -\tfrac{i}{2}A_{0i}(\bar{\psi}\gamma^{0}\gamma^{i}\psi - \bar{\psi}\gamma^{i}\gamma^{0}\psi). \end{aligned} \tag{3.19}$$

When $\mu = i$ and $\nu = 0$:

$$F_{6P} = \frac{i}{2} A_{i0} \bar{\psi} \gamma^0 (\gamma^i \gamma^0 - \gamma^0 \gamma^i) \gamma^0 \psi$$
$$= \tfrac{i}{2} A_{i0} (\bar{\psi} \gamma^0 \gamma^i \psi - \bar{\psi} \gamma^i \gamma^0 \psi) \tag{3.20}$$
$$= -\tfrac{i}{2} A_{0i} (\bar{\psi} \gamma^i \gamma^0 \psi - \bar{\psi} \gamma^0 \gamma^i \psi).$$

For the remaining two cases, parity is odd since F_6 does not remain the same under the transformation.

Under charge conjugation, when μ= ν and can be either 0 or i, it works out to be the trivial case just like above. For the other two cases where $\mu = 0$ and $\nu = i$ or $\mu = i$ and $\nu = 0$,

$$F_{6C} = -\frac{i}{2} A_{\mu\nu} \bar{\psi} \gamma^0 \gamma^2 (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \gamma^0 \gamma^2 \psi$$
$$= -\tfrac{i}{2} A_{\mu\nu} (\bar{\psi} \gamma^0 \gamma^\mu \gamma^0 \gamma^2 \gamma^2 \gamma^\nu \psi - \bar{\psi} \gamma^0 \gamma^\nu \gamma^0 \gamma^2 \gamma^2 \gamma^\mu \psi$$
$$= -\tfrac{i}{2} A_{\mu\nu} \bar{\psi} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \psi \tag{3.21}$$

and both cases will be C odd as we can see. So ignoring the trivial cases, F_6 can only be CP-even, which is not what we desire. Thus, F_6 is eliminated from our choices of modified lagrangians.

It is also worth noting that the CP violation indeed comes from A_0, as mentioned in Section 2.2. As we can see from Table 1, it is only when $\mu = 0$ then we can get the CP odd for all of the modified lagrangians.

Chapter 4

Plane-wave Approximations and Modified Energy Dispersion Relation

In this chapter, we will find the plane-wave solutions to our modified Lagrangians and just as the typical Dirac Lagrangian; the solutions should be simultaneous eigenstates of energy and momentum. In Schrödinger representation, the energy-eigenvalue is as below

$$i\hbar\partial_t\psi_E = E\psi_E. \tag{4.1}$$

Whereas for momentum, $\vec{p} = -ih\vec{\nabla}$ and the eigenvalue of momentum is given by

$$\vec{p}\psi_p = p\psi_p. \tag{4.2}$$

We wish to pursue solutions of the following form

$$\psi(\boldsymbol{x}, t) = e^{-ik.x}u(k) \tag{4.3}$$

where k_μ is a four vector and setting $\hbar = c = 1$ and *u(k)* is the associated bispinor.

4.1 Deriving the Plane-wave Solution

Since the *x* component is confined to the exponent, we have

$$\partial_\mu\psi = -ik_\mu e^{-ik.x}u \tag{4.4}$$

because we assumed the wavefunction to be as per Equation (4.3).

We then substitute this into the Dirac equation and after simplification, we get

$$(\gamma^{\mu}k_{\mu}-m)u=\begin{pmatrix}E-m & -\vec{k}.\vec{\sigma}\\ \vec{k}.\vec{\sigma} & E+m\end{pmatrix}\begin{pmatrix}u_a\\ u_b\end{pmatrix}=\begin{pmatrix}(E-m)u_a-\vec{k}.\vec{\sigma}u_b\\ \vec{k}.\vec{\sigma}u_a-(E+m)u_b\end{pmatrix}=0 \tag{4.5}$$

where u_a represents the upper two components of the bispinor and u_b represents the lower two components and $\vec{\sigma}$ are the Pauli matrices. Also, $k_0=E$ while $\vec{k}=k_i$.

In order to satisfy the condition of Equation (4.5), we can then obtain expressions for u_a and u_b which is as follows,

$$u_a=\frac{\vec{k}.\vec{\sigma}}{E-m}u_b \tag{4.6}$$

$$u_b=\frac{\vec{k}.\vec{\sigma}}{E+m}u_a. \tag{4.7}$$

By substituting Equation (4.6) into Equation (4.7) or vice versa, we get

$$u_a=\frac{\left(\vec{k}.\vec{\sigma}\right)^2}{E^2-m^2}u_a \tag{4.8}$$

Evaluating $\left(\vec{k}.\vec{\sigma}\right)^2$,

$$\vec{k}.\vec{\sigma}=k_x\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}+k_y\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}+k_z\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}=\begin{pmatrix}k_z & (k_x-ik_y)\\ (k_x+ik_y) & -k_z\end{pmatrix} \tag{4.9}$$

$$(\vec{k}.\vec{\sigma})^2=\begin{pmatrix}k_z^2+(k_x-ik_y)(k_x+ik_y) & k_z(k_x-ik_y)-k_z(k_x-ik_y)\\ k_z(k_x+ik_y)-k_z(k_x+ik_y) & (k_x+ik_y)(k_x-ik_y)+k_z^2\end{pmatrix}=\vec{k}^2 1 \tag{4.10}$$

where 1 is the identity matrix. So simplifying Equation (4.8), we will get

$$u_a=\frac{\vec{k}^2}{E^2-m^2}u_a. \tag{4.11}$$

From this, we get back the dispersion relation

$$E^2 - m^2 = \vec{k}^2. \tag{4.12}$$

To obtain the plane wave solution to the Dirac equation, we consider four different cases:

1. Let $u_a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $u_b = \frac{\vec{k}.\vec{\sigma}}{E+m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{E+m} \begin{pmatrix} k_z \\ k_x + ik_y \end{pmatrix}$. The first canonical solution will then be

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{k_z}{E+m} \\ \frac{k_x + ik_y}{E+m} \end{pmatrix}. \tag{4.13}$$

2. Let $u_a = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then $u_b = \frac{\vec{k}.\vec{\sigma}}{E+m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{E+m} \begin{pmatrix} k_x - ik_y \\ -k_z \end{pmatrix}$. The second canonical solution will then be

$$u^{(2)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{k_x - ik_y}{E+m} \\ \frac{-k_z}{E+m} \end{pmatrix}. \tag{4.14}$$

3. Let $u_b = \begin{pmatrix}1\\0\end{pmatrix}$, then $u_a = \frac{\vec{k}.\vec{\sigma}}{E-m}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{E-m}\begin{pmatrix}k_z\\k_x + ik_y\end{pmatrix}$. The third canonical solution will then be

$$u^{(3)} = N\begin{pmatrix}\frac{k_z}{E-m}\\\frac{k_x+ik_y}{E-m}\\1\\0\end{pmatrix}. \tag{4.15}$$

4. Let $u_b = \begin{pmatrix}0\\1\end{pmatrix}$, then $u_a = \frac{\vec{k}.\vec{\sigma}}{E-m}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{E-m}\begin{pmatrix}k_x - ik_y\\-k_z\end{pmatrix}$. The fourth canonical solution will then be

$$u^{(4)} = N\begin{pmatrix}\frac{k_x-ik_y}{E-m}\\\frac{-k_z}{E-m}\\0\\1\end{pmatrix}. \tag{4.16}$$

Where N is normalization factor, $\sqrt{E+m}$. [5]

4.2 Modified Energy Dispersion Relation (MDR)

From our modified Lagrangians, we are going to apply the plane wave solution to get the MDRs. Continuing to use F_3 as an example; we apply it to the Euler-Lagrange equation,

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \bar{\psi}\right)}\right) = 0 \tag{4.17}$$

and solve it to obtain the desired MDR.

Referring to Equation (2.6),

$$\frac{\partial \mathcal{L}_3}{\partial \psi} = i\gamma^{\mu}\partial_{\mu}\psi - m\psi - iC_{\mu}(\partial_{\mu}\psi) \tag{4.18}$$

$$\frac{\partial \mathcal{L}_3}{\partial(\partial_{\mu}\bar{\psi})} = iC_{\mu}\psi$$

$$\Rightarrow \partial_{\mu}\left(\frac{\partial \mathcal{L}_3}{\partial(\partial_{\mu}\bar{\psi})}\right) = i\partial_{\mu}C_{\mu}\psi - iC_{\mu}(\partial_{\mu}\psi) = iC_{\mu}(\partial_{\mu}\psi) \tag{4.19}$$

since $i\partial_{\mu}C_{\mu}\psi = 0$. Substituting Equations (4.18) and (4.19) into the Euler-Lagrange equation:

$$i\gamma^{\mu}\partial_{\mu}\psi - m\psi - 2iC_{\mu}(\partial_{\mu}\psi) = 0. \tag{4.20}$$

We know that $k_{\mu} = i\partial_{\mu}$ (setting h=1) and from the Dirac equation $(i\gamma^{\mu}\partial_{\mu}\psi - m\psi = 0)$, we can obtain the following equation:

$$\begin{aligned} &i\gamma^{\mu}\partial_{\mu}\psi = m\psi \\ &\Rightarrow i(\gamma^{\mu})^2\partial_{\mu}\psi = \gamma^{\mu}m\psi \\ &\Rightarrow i\partial_{\mu}\psi = \gamma^{\mu}m\psi \end{aligned} \tag{4.21}$$

where $(\gamma^{\mu})^2 = 1$. We can then apply these substitutions into Equation (4.20) and it will become

$$\begin{aligned} &\gamma^{\mu}k_{\mu}\psi - m\psi - 2C_{\mu}\gamma^{\mu}m\psi = 0 \\ &\Rightarrow [\gamma^{\mu}(k_{\mu} - 2C_{\mu}m) - m]\psi = 0. \end{aligned} \tag{4.22}$$

Squaring both terms, we will obtain the following expression

$$\left(k_\mu - 2C_\mu m\right)^2 - m^2 = 0$$
$$k_\mu{}^2 + 4C^2m^2 - 4k_\mu C_\mu m - m^2 = 0$$
$$\boldsymbol{k}^2 = m^2 - 4C^2m^2 + 4\boldsymbol{C}.\boldsymbol{k}m \tag{4.23}$$

which is the MDR for $\mathcal{L}_3$ that we are seeking. Here, $\boldsymbol{k}^2 = E^2 - \vec{p}^2$ and all boldfaced letters represent 4-vectors.

The exact method is applied to the rest of the Lagrangians and below is a summary of the different MDRs found.

For $\mathcal{L}_1$:

$$\boldsymbol{k}^2 = m^2 - \boldsymbol{A}^2 - 2\boldsymbol{k}.\boldsymbol{A}. \tag{4.24}$$

For $\mathcal{L}_2$:

$$\boldsymbol{k}^2\left(\boldsymbol{k}^2 - 2m^2 + 2B_0^2 - 2B_i^2\right)$$
$$= 4E^2B_0^2 + 4k_i{}^2B_i^2 + 8B_0k_iB_i - \boldsymbol{B}^4 - m^4 - 2B_0^2B_i^2 - 2B_0^2m^2 - 6B_i^2m^2 \tag{4.25}$$

which may look rather complicated now, but it will simplify when we apply the condition that $\mu = 0$ for F_2.

For $\mathcal{L}_3$:

$$\boldsymbol{k}^2 = m^2 - 4\boldsymbol{C}^2m^2 + 4\boldsymbol{C}.\boldsymbol{k}m. \tag{4.26}$$

For $\mathcal{L}_4$:

$$\boldsymbol{k}^2 = m^2 + 4\boldsymbol{D}^2m^2 + 4\boldsymbol{D}m^2. \tag{4.27}$$

For $\mathcal{L}_5$:

$$\boldsymbol{k}^2 = \frac{m^2\left[16E^2\, G_0^2 - m^2 + 8m^2(G_0^2 - G_i^2) - 16m^2 G_0^4 + 32m^2 G_0^2 G_i^2\right]}{\boldsymbol{k}^2 - 2m^2 + 8m^2(G_0^2 - G_i^2)}. \tag{4.28}$$

As mentioned, the condition based on discrete symmetries as derived in the previous section has not been applied yet, we will do that in the next segment. Detailed derivations of the MDRs can be found in **Appendix B**.

Chapter 5

Neutrino Oscillations

Neutrinos are active research of interest in recent years because once we fully comprehend the mechanisms of neutrinos; we can probe into new physics that is still unbeknownst to us. First predicted by Bruno Pontecorvo in 1957, neutrinos are observed to change its flavour and this phenomenology is known as neutrino oscillations. The orthodox reasoning behind this phenomenology is that neutrinos have mass; contrary to what the Standard Model suggested. The presence of a Lorentz –violating background field suggests an alternative explanation of neutrino oscillations.

In this chapter, we will first be looking at the conventional theory behind neutrino oscillation and how the probability of oscillation depends on the energy dispersion relation. Equipped with this knowledge, we can then apply our modified energy dispersion relations and see how the probabilities vary. Thereafter we can approximate the magnitudes of our background fields as we can obtain values of the different parameters collected from experiments. We will then discuss the significance of our results.

5.1 Orthodox Theory

Given that neutrinos have masses, there exists neutrino mass eigenstates ν_i, where i =1, 2, …, each with a mass m_i. To understand leptonic mixing, consider the following leptonic decay of the W boson:

$$W^+ \to \nu_i + \overline{l_\alpha} \tag{5.1}$$

where α = e, μ or τ and l_e, l_μ and l_τ are electron, muon and tau respectively. Now, leptonic mixing simply means that when W^+decays to a certain $\overline{l_\alpha}$, the concomitant neutrino mass eigenstate can be any of the different ν_i. This means that every time a W^+boson decays, the resulting mass eigenstate need not be the same each time. The amplitude for the decay of W^+ to a specific $\nu_i + \overline{l_\alpha}$ is denoted by $U^*_{\alpha i}$, where $U_{\alpha i}$ is a particular element of lepton mixing matrix.

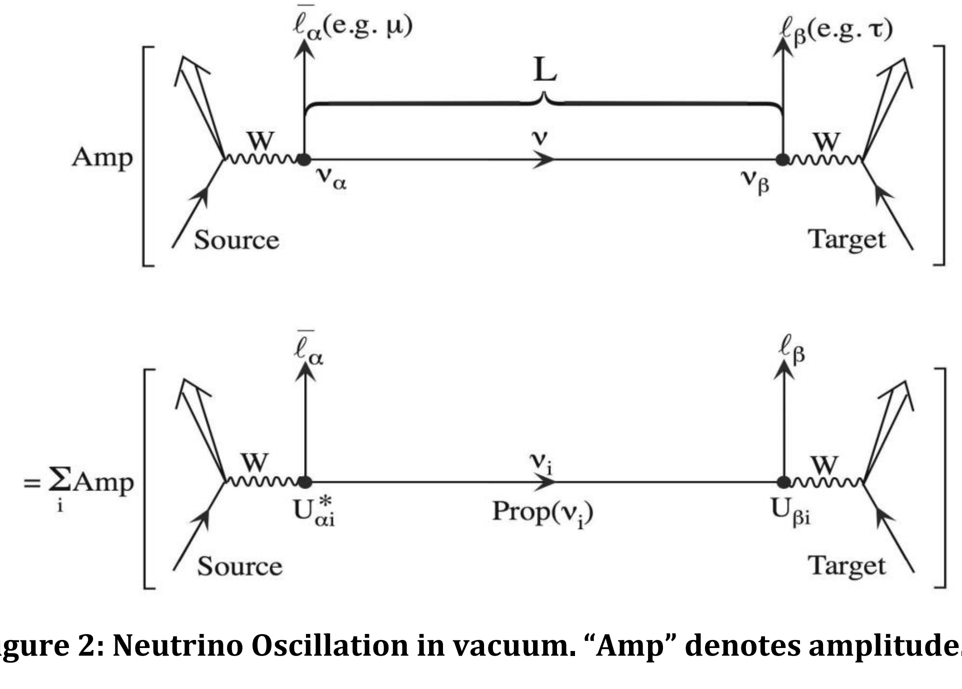

Figure 2: Neutrino Oscillation in vacuum. "Amp" denotes amplitude. [6]

The amplitude of a neutrino undergoing flavour change, say from α to β is composed of three factors, as seen in Figure 2. The first is the amplitude for the neutrino produced to be of a particular ν_i, which is aforementioned to be $U^*_{\alpha i}$. The second is the amplitude for the ν_i produced to travel from the source to the detector and is henceforth denoted as Prop(ν_i). Lastly is the amplitude for the lepton produced by ν_i to be of a particular flavour, denoted by $U_{\beta i}$. Thus the final amplitude is given by:

$$\text{Amp}\,(\nu_\alpha \rightarrow \nu_\beta) = \sum_i\, U^*_{\alpha i}\, \text{Prop}(\nu_i) U_{\beta i}. \tag{5.2}$$

To determine Prop(ν_i), consider the Schrödinger equation of ν_i in its rest frame:

$$i \frac{\partial}{\partial \tau_i} |\nu_i(\tau_i)\rangle = m_i |\nu_i(\tau_i)\rangle \tag{5.3}$$

where τ_i is the time in rest frame while m_i is the rest mass of the neutrino eigenstate.

When solved, it gives us:

$$|\nu_i(\tau_i)\rangle = e^{-i m_i \tau_i} |\nu_i(0)\rangle. \tag{5.4}$$

Hence,

$$\text{Prop}(\nu_i) = e^{-i m_i \tau_i}. \tag{5.5}$$

When expressed in terms of lab frame variables,

$$m_i \tau_i = E_i t - p_i L \tag{5.6}$$

with E_i and p_i being the energy and momentum of $\nu_i$while t and L are the time and distance between the source and the detector.

To contribute coherently to a neutrino oscillation signal, the components of the neutrino beam must be of the same energy, thus we can make the approximation $E_i \approx E$. So assuming $m_i \ll E^2$, the momentum p_i is given by:

$$p_i = \sqrt{E^2 - {m_i}^2} \cong E - \frac{{m_i}^2}{2E}. \tag{5.7}$$

Thus,

$$m_i\tau_i \cong E(t-L) + \frac{{m_i}^2}{2E}L. \tag{5.8}$$

Since the phase $E(t-L)$ is prevalent to all the interfering mass eigenstates, it can be ignored, hence:

$$\mathrm{Prop}(\nu_i) = \exp\left[-i{m_i}^2\frac{L}{2E}\right]. \tag{5.9}$$

For three-neutrino oscillation, we have

$$\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = \begin{pmatrix}U_{e1} & U_{e2} & U_{e3}\\ U_{\mu1} & U_{\mu2} & U_{\mu3}\\ U_{\tau1} & U_{\tau2} & U_{\tau3}\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix}. \tag{5.10}$$

U here is a unitary matrix known as the Pontecorvo-Maka-Nagakawa-Sakata (PMNS) matrix, which is given in **Appendix C**.

Let's assume that at time t=0, a neutrino in a pure $|\nu_\alpha\rangle$ state:

$$|\psi(0)\rangle = U_{\alpha1}|\nu_1\rangle + U_{\alpha2}|\nu_2\rangle + U_{\alpha3}|\nu_3\rangle. \tag{5.11}$$

As it evolves through time,

$$|\psi(t)\rangle = U_{\alpha1}e^{-ip_1.x}|\nu_1\rangle + U_{\alpha2}e^{-ip_2.x}|\nu_2\rangle + U_{\alpha3}e^{-ip_3.x}|\nu_3\rangle \tag{5.12}$$

where $p_i.x = E_i t - \vec{p}.\vec{\mathrm{x}}$.

After propagating through a distance L, the wavefunction becomes:

$$|\psi(L)\rangle = U_{\alpha1}e^{-i\varphi_1}|\nu_1\rangle + U_{\alpha2}e^{-i\varphi_2}|\nu_2\rangle + U_{\alpha3}e^{-i\varphi_3}|\nu_3\rangle. \tag{5.13}$$

We assumed that the neutrino is relativistic, so $\varphi_i = p_i.x = E_i t - |\boldsymbol{p_i}|L \approx (E_i - |\boldsymbol{p_i}|)L$.

From Equation (5.6), we can then approximate E_i to be

$$E_i \approx p_i + \frac{m_i^2}{2E_i}. \tag{5.14}$$

Thus,

$$\varphi_i = (E_i - |p_i|)L \approx \frac{m_i^2}{2E_i}L. \tag{5.15}$$

We then express the mass eigenstate as superposition of the flavour eigenstates:

$$\begin{aligned} |\psi(L)\rangle = &\left(U_{\alpha 1}e^{-i\varphi_1}U_{e1}^* + U_{\alpha 2}e^{-i\varphi_2}U_{e2}^* + U_{\alpha 3}e^{-i\varphi_3}U_{e3}^*\right)|\nu_e\rangle \\ &+\left(U_{\alpha 1}e^{-i\varphi_1}U_{\mu 1}^* + U_{\alpha 2}e^{-i\varphi_2}U_{\mu 2}^* + U_{\alpha 3}e^{-i\varphi_3}U_{\mu 3}^*\right)|\nu_\mu\rangle \\ &+\left(U_{\alpha 1}e^{-i\varphi_1}U_{\tau 1}^* + U_{\alpha 2}e^{-i\varphi_2}U_{\tau 2}^* + U_{\alpha 3}e^{-i\varphi_3}U_{\tau 3}^*\right)|\nu_\tau\rangle. \end{aligned} \tag{5.16}$$

So the oscillation probability in the case of three neutrinos is:

$$\begin{aligned} \mathrm{P}\left(\nu_\alpha \to \nu_\beta\right) &= \left|\,\langle \nu_\beta|\psi(L)\rangle\,\right|^2 \\ &= \left|U_{\alpha 1}e^{-i\varphi_1}U_{\beta 1}^* + U_{\alpha 2}e^{-i\varphi_2}U_{\beta 2}^* + U_{\alpha 3}e^{-i\varphi_3}U_{\beta 3}^*\right|^2. \end{aligned} \tag{5.17}$$

The oscillation probability is different for each of the nine types of flavour change.

In this thesis, we shall focus on the a particular transition probability, that of ν_μ transiting to ν_e.

$$\begin{aligned} \mathrm{P}\left(\nu_\mu \to \nu_e\right) &= |\,\langle \nu_e|\psi(L)\rangle\,|^2 \\ &= \left|U_{\mu 1}e^{-i\varphi_1}U_{e1}^* + U_{\mu 2}e^{-i\varphi_2}U_{e2}^* + U_{\mu 3}e^{-i\varphi_3}U_{e3}^*\right|^2. \end{aligned} \tag{5.18}$$

The reason is that CP violation enters neutrino oscillations through θ_{13} and the experiments that determines the third neutrino mixing angle is known as the accelerator experiments, such as the T2K (Tokai to Kamioka) experiment in Japan that search for appearance of ν_e in ν_μ beams.

Using the following complex relationship:

$$|Z_1 + Z_2 + Z_3|^2 = |Z_1|^2 + |Z_2|^2 + |Z_3|^2 + 2\boldsymbol{Re}(Z_1 Z_2^* + Z_1 Z_3^* + Z_2 Z_3^*). \tag{5.19}$$

Equation (5.18) becomes

$$\begin{aligned}
\mathrm{P}(\nu_\mu \to \nu_e) &= \left|U_{\mu 1}U_{e1}^* e^{-i\varphi_1}\right|^2 + \left|U_{\mu 2}U_{e2}^* e^{-i\varphi_2}\right|^2 + \left|U_{\mu 3}U_{e3}^* e^{-i\varphi_3}\right|^2 \\
&\quad +2\boldsymbol{Re}(U_{\mu 1}U_{e1}^* e^{-i\varphi_1} U_{\mu 2}^* U_{e2} e^{i\varphi_2} + U_{\mu 1}U_{e1}^* e^{-i\varphi_1} U_{\mu 3}^* U_{e3} e^{i\varphi_3} \\
&\quad +U_{\mu 2}U_{e2}^* e^{-i\varphi_2} U_{\mu 3}^* U_{e3} e^{i\varphi_3}) \\
&= -2\boldsymbol{Re}\left(U_{\mu 1}U_{e1}^* U_{\mu 2}^* U_{e2} + U_{\mu 1}U_{e1}^* U_{\mu 3}^* U_{e3} + U_{\mu 2}U_{e2}^* U_{\mu 3}^* U_{e3}\right) \\
&\quad +2\boldsymbol{Re}[U_{\mu 1}U_{e1}^* e^{-i(\varphi_1-\varphi_2)} U_{\mu 2}^* U_{e2} + U_{\mu 1}U_{e1}^* e^{-i(\varphi_1-\varphi_3)} U_{\mu 3}^* U_{e3} \\
&\quad +U_{\mu 2}U_{e2}^* e^{-i(\varphi_2-\varphi_3)} U_{\mu 3}^* U_{e3}] \\
&= 2\boldsymbol{Re}\{U_{\mu 1}U_{e1}^* U_{\mu 2}^* U_{e2}\left[e^{i(\varphi_2-\varphi_1)} - 1\right] + U_{\mu 1}U_{e1}^* U_{\mu 3}^* U_{e3}\left[e^{i(\varphi_3-\varphi_1)} - 1\right] \\
&\quad +U_{\mu 2}U_{e2}^* U_{\mu 3}^* U_{e3}\left[e^{i(\varphi_3-\varphi_2)} - 1\right]\} \\
&= 2\boldsymbol{Re}\{U_{\mu 1}U_{e1}^* U_{\mu 2}^* U_{e2}[\cos(\varphi_2 - \varphi_1) - 1] \\
&\qquad +U_{\mu 1}U_{e1}^* U_{\mu 3}^* U_{e3}[\cos(\varphi_3 - \varphi_1) - 1] \\
&\qquad +U_{\mu 2}U_{e2}^* U_{\mu 3}^* U_{e3}[\cos(\varphi_3 - \varphi_2) - 1]\} \\
&= -4\boldsymbol{Re}[U_{\mu 1}U_{e1}^* U_{\mu 2}^* U_{e2}\sin^2\left(\Delta m_{21}^2 \frac{L}{4E}\right) + U_{\mu 1}U_{e1}^* U_{\mu 3}^* U_{e3}\sin^2(\Delta m_{31}^2 \frac{L}{4E}) \\
&\qquad +U_{\mu 2}U_{e2}^* U_{\mu 3}^* U_{e3}\sin^2(\Delta m_{32}^2 \frac{L}{4E})]
\end{aligned} \tag{5.20}$$

where

$$\varphi_i - \varphi_j = \frac{\Delta m_{ij}^2}{2E} L. \tag{5.21}$$

For expansion of the *Real* parts of the terms, refer to **Appendix D**.

We can see from Equation (5.20) that the oscillation probability is dependent on the mass squared difference; there will be no oscillation if Δm_{ij}^2 is zero. So from the evidence that neutrinos indeed oscillate, it is implied that neutrinos are not massless as we thought them to be. Another important observation is that up till now, we cannot determine the exact mass of the neutrinos, we can only do with knowing the mass squared difference between the different mass eigenstates for now.

5.2 Neutrino Oscillation Probability for MDRs

The usual dispersion relation used in the conventional theory is known to be

$$E^2 - {p_i}^2 = m^2 \tag{5.22}$$

after setting $c = 1$. As we can see from the previous section, the dispersion relation affects the oscillation probability directly by entering through the factor φ_i. Thus with an altered dispersion relation, a change in oscillation probability is expected. We continue to use $\mathcal{L}_3$ as an example here.

Starting with Equation (4.26), it becomes

$$E^2 - {p_i}^2 = m^2 - 4C^2m^2 + 4m(C_0E + C_i.\vec{p}) \tag{5.23}$$

after expanding the four-momentum.

Since μ must be equivalent to zero in order to have CP violation, we have

$${p_i}^2 = E^2 - m^2 + 4{C_0}^2m^2 - 4mC_0E. \tag{5.24}$$

Equation (5.6) now becomes

$$
\begin{aligned}
p_i &= \sqrt{E^2 - {m_i}^2 + 4{C_0}^2{m_i}^2 - 4m_i{C_0}_iE} \\
&= E\left(1 - \frac{{m_i}^2}{E^2} + \frac{4{C_0}^2{m_i}^2}{E^2} - \frac{4m_i{C_0}_i}{E}\right)^{\frac{1}{2}} \\
&\approx E(1 - \frac{1}{2}(\frac{{m_i}^2}{E^2} - \frac{4{C_0}^2{m_i}^2}{E^2} + \frac{4m_i{C_0}_i}{E}) \\
&\approx E(1 - \frac{{m_i}^2}{2E^2} - \frac{2m_i{C_0}_i}{E})
\end{aligned} \tag{5.25}
$$

after doing a Taylor expansion and neglecting higher order terms as we assumed C_o is small. Equation (5.15) then becomes

$$
\begin{aligned}
\varphi_i &= EL - E(1 - \frac{{m_i}^2}{2E^2} + \frac{2m_i{C_0}_i}{E})L \\
&= (\frac{{m_i}^2}{2E^2} - 2m_i{C_0}_i)L.
\end{aligned} \tag{5.26}
$$

The probability of neutrino to change its flavour from μ to e is then

$$
\begin{aligned}
\mathrm{P}(\nu_\mu \to \nu_e) = -4\boldsymbol{Re}\{&U_{\mu 1}U_{e1}^*U_{\mu 2}^*U_{e2}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{21}^2}{2E} - 2\Delta m_{21}\Delta {C_0}_{21}\right)\right] \\
&+U_{\mu 1}U_{e1}^*U_{\mu 3}^*U_{e3}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{31}^2}{2E} - 2\Delta m_{31}\Delta {C_0}_{31}\right)\right] \\
&+U_{\mu 2}U_{e2}^*U_{\mu 3}^*U_{e3}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{32}^2}{2E} - 2\Delta m_{32}\Delta {C_0}_{32}\right)\right]\}
\end{aligned} \tag{5.27}
$$

where $\Delta {C_0}_{ij} = {C_0}_i - {C_0}_j$ is the difference between the interactions of the Lorentz-violating background field with the different mass eigenstates, from the assumption that the different mass eigenstates interact uniquely with the background field.

To determine the oscillation probabilities for the rest of the MDRs, the same method is applied and the results are summarized as below. Full details will be shown in **Appendix E**.

For $\mathcal{L}_1$:

$$\mathrm{P}(\nu_\mu \to \nu_e) = -4\boldsymbol{Re}\{U_{\mu 1}U_{e1}^*U_{\mu 2}^*U_{e2}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{21}^2}{2E} - \Delta A_{0_{21}}\right)\right]$$
$$+U_{\mu 1}U_{e1}^*U_{\mu 3}^*U_{e3}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{31}^2}{2E} - \Delta A_{0_{31}}\right)\right]$$
$$+U_{\mu 2}U_{e2}^*U_{\mu 3}^*U_{e3}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{32}^2}{2E} - \Delta A_{0_{32}}\right)\right]\}. \quad (5.28)$$

For $\mathcal{L}_2$:

$$\mathrm{P}(\nu_\mu \to \nu_e) = -4\boldsymbol{Re}\{U_{\mu 1}U_{e1}^*U_{\mu 2}^*U_{e2}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{21}^2}{2E} \pm \Delta B_{0_{21}}\right)\right]$$
$$+U_{\mu 1}U_{e1}^*U_{\mu 3}^*U_{e3}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{31}^2}{2E} \pm \Delta B_{0_{31}}\right)\right]$$
$$+U_{\mu 2}U_{e2}^*U_{\mu 3}^*U_{e3}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{32}^2}{2E} \pm \Delta B_{0_{32}}\right)\right]\}. \quad (5.29)$$

For $\mathcal{L}_3$:

$$\mathrm{P}(\nu_\mu \to \nu_e) = -4\boldsymbol{Re}\{U_{\mu 1}U_{e1}^*U_{\mu 2}^*U_{e2}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{21}^2}{2E} - 2\Delta m_{21}\Delta C_{0_{21}}\right)\right]$$
$$+U_{\mu 1}U_{e1}^*U_{\mu 3}^*U_{e3}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{31}^2}{2E} - 2\Delta m_{31}\Delta C_{0_{31}}\right)\right]$$
$$+U_{\mu 2}U_{e2}^*U_{\mu 3}^*U_{e3}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{32}^2}{2E} - 2\Delta m_{32}\Delta C_{0_{32}}\right)\right]\}. \quad (5.30)$$

For $\mathcal{L}_4$:

$$\mathrm{P}(\nu_\mu \to \nu_e) = -4\boldsymbol{Re}\{U_{\mu 1}U_{e1}^*U_{\mu 2}^*U_{e2}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{21}^2}{2E}(1 + 4\Delta D_{0_{21}})\right)\right]$$
$$+U_{\mu 1}U_{e1}^*U_{\mu 3}^*U_{e3}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{31}^2}{2E}(1 + 4\Delta D_{0_{21}})\right)\right]$$
$$+U_{\mu 2}U_{e2}^*U_{\mu 3}^*U_{e3}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{32}^2}{2E}(1 + 4\Delta D_{0_{21}})\right)\right]\}. \quad (5.31)$$

For $\mathcal{L}_5$:

$$\mathrm{P}(\nu_\mu \to \nu_e) = -4\boldsymbol{Re}\{U_{\mu 1}U_{e1}^{*}U_{\mu 2}^{*}U_{e2}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{21}^2}{2E} \pm 2\Delta m_{21}\Delta G_{0_{21}}\right)\right]$$
$$+U_{\mu 1}U_{e1}^{*}U_{\mu 3}^{*}U_{e3}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{31}^2}{2E} \pm 2\Delta m_{31}\Delta G_{0_{31}}\right)\right]$$
$$+U_{\mu 2}U_{e2}^{*}U_{\mu 3}^{*}U_{e3}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{32}^2}{2E} \pm 2\Delta m_{32}\Delta G_{0_{32}}\right)\right]\}. \quad (5.32)$$

As one can see, the oscillation probability defers from that of the conventional neutrino oscillation phenomenology; with the difference stemming from the Lorentz-violating background field. We now wish to ascertain the bounds of these background fields and this will be shown in the next section.

5.3 Determination of the Magnitude of the Background Fields

From our oscillation probabilities, it is possible to give estimated values of the order of magnitude of the background fields and this will be the focus of this section. The background field is still not observed until now, but its value should be within the error bar of the experimental results. Thus we can approximate the error of the first term $\left(\frac{\Delta m_{ij}^2}{2E}\right)$ in the parenthesis of the $\sin^2$ term of the probabilities to be of the same order of magnitude as the second term (that contains the ΔA term). The values of the known parameters (i.e., Δm_{ij}^2 and E) are taken from experimental data. The values of the respective parameters are shown in Table 2.

Parameter	Value	Experiments that Measured Parameters
$\Delta m_{sol}{}^2 = \lvert \Delta m_{21}^2 \rvert$	$(7.53 \pm 0.18) \times 10^{-5} eV^2$	Long baseline reactor neutrino experiment, eg. Kamland, Super-Kamiokande, Sudbury.
$\delta m_{21} \approx \sqrt{\delta(\Delta m_{21}^2)}$	$1.34 \times 10^{-3}\mathrm{eV}$	
$\Delta m_{atm}{}^2 = \lvert \Delta m_{32}^2 \rvert \approx \lvert \Delta m_{31}^2 \rvert$	$(2.44 \pm 0.06) \times 10^{-3} eV^2$	Atmospheric and long baseline accelerator neutrino oscillation experiments, eg. MINOS/K2K.
$\delta m_{31} \approx \delta m_{32} \approx \sqrt{\delta(\Delta m_{32}^2)}$	$7.75 \times 10^{-3}\mathrm{eV}$	
E	0.6GeV	Energy of neutrino beam in T2K experiment [9]

Table 2: Experiment Values of the Different Parameters. [8]

As we can see from Table 2, we made the assumption that error of the mass difference of neutrino is approximately the square root of the error of the mass squared difference

$$m_{ij} \approx \sqrt{\delta\left(\Delta m_{\mathrm{ij}}^2\right)}. \tag{5.33}$$

It is observed that from these five Lagrangians, we only have four distinct value of order of the magnitude of the background fields because $\mathcal{L}_1$ and $\mathcal{L}_2$ will give the same value. This will be shown in the upcoming part. The values calculated are after restoring ħ and c and are dimensionless.

For $\mathcal{L}_1$ and $\mathcal{L}_2$:

From Equation (5.28), to determine the magnitude of the difference in background field, we do the following approximation

$$\text{Magnitude of } \Delta A_{0_{12}} \approx 10^{10} \tag{5.34}$$

$$\text{Magnitude of } \Delta A_{0_{13}} \approx \text{Magnitude of } \Delta A_{0_{23}} \approx 10^{12}. \tag{5.35}$$

Similarly from Equation (5.29),

$$\text{Magnitude of } \Delta B_{0_{12}} \approx 10^{10} \tag{5.36}$$

$$\text{Magnitude of } \Delta B_{0_{13}} \approx \text{Magnitude of } \Delta B_{0_{23}} \approx 10^{12}. \tag{5.37}$$

The magnitudes of the background fields in $\mathcal{L}_1$ and $\mathcal{L}_2$ are the same. The units of these background fields are $\frac{[E]}{[\hbar c]}$.

***For $\mathcal{L}_3$*:**

Just as before, we approximate the terms in the parenthesis of $\sin^2$ from Equation (5.30) to be of the same order of magnitude.

$$\text{Magnitude of } \Delta C_{0_{12}} \approx 10^{-29}. \tag{5.38}$$

$$\text{Magnitude of } \Delta C_{0_{13}} \approx \text{Magnitude of } \Delta C_{0_{23}} \approx 10^{-28}. \tag{5.39}$$

For $\mathcal{L}_4$:

From Equation (5.31), the terms in the approximation are observed to be different from the other Lagrangians, because both terms in the parenthesis share the same factor of $\frac{\Delta m_{ij}^2}{2E}$, in addition to $\frac{L}{2}$ and so the magnitude of the background field is different as compared to the rest.

$$\text{Magnitude of } \Delta D_{0_{12}} \approx \text{Magnitude of } \Delta D_{0_{13}} \approx \text{Magnitude of } \Delta D_{0_{23}} \approx 10^{-34}. \quad (5.40)$$

For $\mathcal{L}_5$:

Similarly from Equation (5.32),

$$\text{Magnitude of } \Delta G_{0_{12}} \approx 10^{-46} \quad (5.41)$$

$$\text{Magnitude of } \Delta G_{0_{13}} \approx \text{Magnitude of } \Delta G_{0_{23}} \approx 10^{-45}. \quad (5.42)$$

For $\mathcal{L}_3$, $\mathcal{L}_4$ and $\mathcal{L}_5$the units of their background fields are $\frac{[E]}{[\mathrm{mc}]}$.

5.4 Discussion of Results

As a matter of fact, these magnitudes calculated are actually the upper bounds of what the background fields should be. One may wonder why there are different values for one particular background field. The reason for the different values is that the separate Lagrangians represent different physics that are still unknown to us, and so the background field interacts differently with each. For these two pairs of Lagrangians that give the same upper bound for the background field, the interpretation is that we cannot distinguish $\mathcal{L}_1$ and $\mathcal{L}_2$ using neutrino oscillations. We have to look into other ways if we were to differentiate between them.

These values will be useful when there are experimental data that differs from the usual. In the lagrangians that we have generated, CP violation is already imposed. So when we have experimental data that we suspect the involvement of CP violation, we can compare these data with our results and see if the potential background field from the data lie within our expected range. If they do, then we have a possible explanation as to where the CP violation effects come from. While comparing the background fields, it is better to convert the background fields from the data collected to dimensionless quantities since the units may be different.

While determining the magnitude of the background field for $\mathcal{L}_3$ and $\mathcal{L}_5$, we notice that there is an additional factor of Δm_{ij} in the probabilities. This is an area of interest because if we have the relevant experimental data, we can actually use this term to determine the mass of the neutrinos. In other words, this term is sensitive to the individual neutrino mass; which is an advantage because up till now, physicists can only determine the mass squared difference of the neutrino mass (as mentioned in the last part of Section 5.1).

Also, from the oscillation probabilities of $\mathcal{L}_1$ and $\mathcal{L}_2$, the background field can be suggested as an alternative explanation to neutrino oscillations. This is because if we were to assume that neutrinos are massless, the oscillations will be due to the interactions with the background field. Hence, whereas the conventional theory of neutrino oscillation provides the evidence that neutrinos are not massless; this alternative, unorthodox theory of neutrino oscillations can support the view that neutrinos can be massless.

Chapter 6

Conclusion

In this thesis, the objective is to obtain the CP-violating effects in neutrino oscillations. To achieve this, we first obtain CP-violating Lagrangians through the imposition of constraints such as hermiticity, locality and universality, which are characteristic of the typical Dirac Lagrangian. But in our case, there is an additional condition under symmetry transformation, which is that the modified Lagrangians must violate CP symmetry. This CP violation enters through the Lorentz-violating background field that we attached to our lagrangians. We then obtain five modified Lagrangians satisfying these conditions; instead of six as we have planned because we found out that one of them did not satisfy the condition of CP violation.

Thereafter, we derived the energy dispersion relations for each Lagrangians using plane wave solutions. These modified energy dispersion relations are then applied to neutrino oscillations and we study how the oscillation probabilities change. From these probabilities, we can determine the magnitude of the background field by approximating specific terms to be of the same order of magnitude; together with the values of the various parameters obtained from experiments, we can give a numerical value of these magnitudes.

In actual fact, these magnitudes give us the upper bound on what the background field should be, if they were observed. From our calculations, we found out that from our five modified Lagrangians, we obtained four specific upper bounds, with a pair of them giving the same value. Despite having the same background field, the Lagrangians actually represent different, new physics. Their interactions with the background field will be unique and hence resulting in the different bounds obtained.

The results that we have gathered can be useful in future as we can compare experimental data, with our results and see if the potential background field from the data lie within our expected range. If it happens to be the case, we can suggest possible explanation as to where the CP violation originated. Also, we have a term that is sensitive to individual neutrino mass, which is an advantage as we can probe the neutrino mass with it; rather than just determining the mass squared difference, which is the limit now. Additionally, we can provide an alternative explanation for neutrino oscillations. That is, it is due to the interaction with the background field that resulted in neutrino oscillation, instead of the mainstream argument in which neutrinos have mass.

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Appendix A: Transformation of Other Modified Lagrangians under Discrete Symmetries

For F_1:

Under parity transformation,

$$F_{1P} = A_\mu \bar{\psi}' \gamma^\mu \psi' = A_\mu \bar{\psi} \gamma^0 \gamma^\mu \gamma^0 \psi. \tag{A.1}$$

When $\mu = 0$:

$$\begin{aligned} F_{1P} &= A_0 \bar{\psi}' \gamma^0 \psi' \\ &= A_0 \bar{\psi} \gamma^0 \gamma^0 \gamma^0 \psi \\ &= A_0 \bar{\psi} \gamma^0 \psi. \end{aligned} \tag{A.2}$$

Since $(\gamma^0)^2 = 1$. We see that when $\mu = 0$, F_1 is P even.

When $\mu = i$:

$$\begin{aligned} F_{1P} &= A_i \bar{\psi}' \gamma^i \psi' \\ &= A_i \bar{\psi} \gamma^0 \gamma^i \gamma^0 \psi \\ &= -A_i \bar{\psi} \gamma^i \psi. \end{aligned} \tag{A.3}$$

We see that when $\mu = i$, F_1 is P odd.

Under charge conjugation,

$$\begin{aligned} F_{1C} &= A_\mu \bar{\psi}' \gamma^\mu \psi' \\ &= A_\mu (-i\gamma^2 \psi \gamma^0) \gamma^\mu (i\gamma^2 \psi^+) \\ &= A_\mu (-i)(\gamma^0 \gamma^2 \psi)^T \gamma^\mu (i)(\bar{\psi} \gamma^0 \gamma^2)^T \\ &= A_\mu (\bar{\psi} \gamma^0 \gamma^2) \gamma^\mu (\gamma^0 \gamma^2 \psi). \end{aligned} \tag{A.4}$$

Since $-i\gamma^2 \psi \gamma^0 = -i(\gamma^2 \psi)^T \gamma^0 = -i(\gamma^0 \gamma^2 \psi)^T$ and $i\gamma^2 \psi^+ = i\gamma^2 \gamma^0 \gamma^0 \psi^+ = i\gamma^2 \gamma^0 \bar{\psi} = i(\psi \gamma^0 \gamma^2)^T$.

When $\mu = 0$:

$$F_{1C} = A_0 \bar{\psi} \gamma^0 \gamma^2 \gamma^0 \gamma^0 \gamma^2 \psi = -A_0 \bar{\psi} \gamma^0 \psi. \tag{A.5}$$

We see that when $\mu = 0$, F_1 is C odd.

When $\mu = i$:

$$F_{1C} = A_i \bar{\psi} \gamma^0 \gamma^2 \gamma^i \gamma^0 \gamma^2 \psi = A_i \bar{\psi} \gamma^i \psi. \tag{A.6}$$

We see that when $\mu = i$, F_1 is C even.

Under time reversal,

$$\begin{aligned} F_{1T} &= A_\mu \bar{\psi}' \gamma^\mu \psi' \\ &= A_\mu (-i\gamma^1 \gamma^3 \bar{\psi}) \gamma^\mu (i\gamma^1 \gamma^3 \psi) \\ &= A_\mu \gamma^1 \gamma^3 \bar{\psi} \gamma^\mu \gamma^1 \gamma^3 \psi. \end{aligned} \tag{A.7}$$

When $\mu = 0$:

$$F_{1T} = A_0 \gamma^1 \gamma^3 \bar{\psi} \gamma^0 \gamma^1 \gamma^3 \psi = A_0 \bar{\psi} \gamma^0 \psi. \tag{A.8}$$

We see that when $\mu = 0$, $F_1$is T odd.

When $\mu = i$:

$$F_{1T} = A_i \gamma^1 \gamma^3 \bar{\psi} \gamma^i \gamma^1 \gamma^3 \psi = -A_i \bar{\psi} \gamma^i \psi. \tag{A.9}$$

We see that when $\mu = i$, $F_1$is T even.

Since we want CP odd, the final form of F_1 is given as such

$$F_1 = A_0 \bar{\psi} \gamma^0 \psi. \tag{A.10}$$

For $\boldsymbol{F_2}$:

Under parity transformation,

$$F_{2P} = A_\mu \bar{\psi}' \gamma^\mu \gamma^5 \psi' = A_\mu \bar{\psi} \gamma^0 \gamma^\mu \gamma^5 \gamma^0 \psi. \tag{A.11}$$

When $\mu = 0$:

$$\begin{aligned} F_{2P} &= B_0 \bar{\psi}' \gamma^0 \gamma^5 \psi' \\ &= B_0 \bar{\psi} \gamma^0 \gamma^0 \gamma^5 \gamma^0 \psi \\ &= -B_0 \bar{\psi} \gamma^0 \gamma^5 \psi. \end{aligned} \tag{A.12}$$

Since $(\gamma^0)^2 = 1$. We see that when $\mu = 0$, F_2 is P odd.

When $\mu = i$:

$$\begin{aligned} F_{2P} &= B_i \bar{\psi}' \gamma^i \gamma^5 \psi' \\ &= B_i \bar{\psi} \gamma^0 \gamma^i \gamma^5 \gamma^0 \psi \\ &= B_i \bar{\psi} \gamma^i \gamma^5 \psi. \end{aligned} \tag{A.13}$$

We see that when $\mu = i$, F_1 is P even.

Under charge conjugation,

$$\begin{aligned} F_{2C} &= B_\mu \bar{\psi}' \gamma^\mu \gamma^5 \psi' \\ &= B_\mu (\bar{\psi} \gamma^2 \gamma^0) \gamma^\mu \gamma^5 (\gamma^0 \gamma^2 \psi) \\ &= B_\mu \bar{\psi} \gamma^\mu \gamma^5 \psi. \end{aligned} \tag{A.14}$$

C is even for F_2.

Under time reversal,

$$F_{2T} = B_\mu \bar{\psi}' \gamma^\mu \gamma^5 \psi' = B_\mu \bar{\psi} \gamma^1 \gamma^3 \gamma^\mu \gamma^5 \gamma^1 \gamma^3 \psi. \tag{A.15}$$

When $\mu = 0$:

$$F_{2T} = B_0\gamma^1\gamma^3\bar{\psi}\gamma^0\gamma^5\gamma^1\gamma^3\psi = B_0\bar{\psi}\gamma^0\gamma^5\psi. \tag{A.16}$$

We see that when $\mu = 0$, $F_2$is T even.

When $\mu = i$:

$$F_{2T} = B_i\gamma^1\gamma^3\bar{\psi}\gamma^i\gamma^5\gamma^1\gamma^3\psi = -B_i\bar{\psi}\gamma^i\gamma^5\psi. \tag{A.17}$$

We see that when $\mu = i$, $F_1$is T odd.

Since we want CP odd, the final form of F_2 is given as such

$$F_2 = B_0\bar{\psi}\gamma^0\gamma^5\psi. \tag{A.18}$$

For F_4:

Under parity transformation,

$$F_{4P} = iD_\mu\gamma^\mu\left(\partial'_\mu\bar{\psi}'\right)\psi' - iC_\mu\gamma^\mu\bar{\psi}'\left(\partial'_\mu\psi'\right). \tag{A.19}$$

When $\mu = 0$:

$$\begin{aligned} F_{4P} &= iD_0\gamma^0(\partial_0\bar{\psi}\gamma^0)\gamma^0\psi - iD_0\gamma^0\bar{\psi}\gamma^0(\partial_0\gamma^0\psi) \\ &= iD_0\gamma^0(\partial_0\bar{\psi})\psi - iD_0\gamma^0\bar{\psi}(\partial_0\psi). \end{aligned} \tag{A.20}$$

We see that when $\mu = 0$, $F_4$is P even.

When $\mu = i$:

$$\begin{aligned} F_{4P} &= -iD_i\gamma^i(\partial_i\bar{\psi}\gamma^0)\gamma^0\psi + iD_i\gamma^i\bar{\psi}\gamma^0(\partial_i\gamma^0\psi) \\ &= -iD_i\gamma^i(\partial_i\bar{\psi})\psi + iD_i\gamma^i\bar{\psi}(\partial_i\psi). \end{aligned} \tag{A.21}$$

We see that when $\mu = i$, $F_4$is P odd.

Under charge conjugation,

$$
\begin{aligned}
F_{4C} &= -iD_\mu\gamma^\mu\left(\partial'_\mu\bar{\psi}'\right)\psi' + iD_\mu\gamma^\mu\bar{\psi}'(\partial_\mu{}'\psi') \\
&= -iD_\mu\gamma^\mu\left(\partial_\mu\bar{\psi}\gamma^0\gamma^2\right)\gamma^0\gamma^2\psi + iD_\mu\gamma^\mu\bar{\psi}\gamma^0\gamma^2(\partial_\mu\gamma^0\gamma^2\psi) \\
&= -iD_\mu\gamma^\mu\left(\partial_\mu\ \bar{\psi}\right)\psi + iD_\mu\gamma^\mu\bar{\psi}(\partial_\mu\psi). \qquad \text{(A.22)}
\end{aligned}
$$

$F_4$is C odd.

Under time reversal,

When $\mu = 0$:

$$
\begin{aligned}
F_{4T} &= -iD_0\gamma^0(\partial_0\bar{\psi}\gamma^1\gamma^3 i)\gamma^1\gamma^3 i\psi + iD_0\gamma^0\bar{\psi}\gamma^1\gamma^3 i(\partial_0\gamma^1\gamma^3 i\psi) \\
&= -iD_0\gamma^0(\partial_0\bar{\psi})\psi + iD_0\gamma^0\bar{\psi}(\partial_0\psi). \qquad \text{(A.23)}
\end{aligned}
$$

We see that when $\mu = 0$, $F_4$is T odd.

When $\mu = i$:

$$
\begin{aligned}
F_{4T} &= -iD_i\gamma^i(\partial_i\bar{\psi}\gamma^1\gamma^3 i)\gamma^1\gamma^3 i\psi + iD_i\gamma^i\bar{\psi}\gamma^1\gamma^3 i(\partial_i\gamma^1\gamma^3 i\psi) \\
&= iD_i\gamma^i(\partial_i\bar{\psi})\psi - iD_i\gamma^i\bar{\psi}(\partial_i\psi). \qquad \text{(A.24)}
\end{aligned}
$$

We see that when $\mu = i$, $F_4$is T even.

Since we want CP odd, the final form of F_4 is given as such

$$
F_4 = iD_0\gamma^0(\partial_0\bar{\psi})\psi - iD_0\gamma^0\bar{\psi}(\partial_0\psi). \qquad \text{(A.25)}
$$

For F_5:

Under parity transformation,

$$F_{5P} = iG_\mu\gamma^5\left(\partial_\mu'\bar{\psi}'\right)\psi' - iG_\mu\gamma^5\bar{\psi}'(\partial_\mu{}'\psi'). \tag{A.26}$$

When $\mu = 0$:

$$\begin{aligned} F_{5P} &= iG_0\gamma^5(\partial_0\bar{\psi}\gamma^0)\gamma^0\psi - iG_0\gamma^5\bar{\psi}\gamma^0(\partial_0\gamma^0\psi) \\ &= iG_0\gamma^5(\partial_0\bar{\psi})\psi - iG_0\gamma^5\bar{\psi}(\partial_0\psi). \end{aligned} \tag{A.27}$$

We see that when $\mu = 0$, $F_5$is P even.

When $\mu = i$:

$$\begin{aligned} F_{5P} &= iG_i\gamma^5(\partial_i\bar{\psi}\gamma^0)\gamma^0\psi - iG_i\gamma^5\bar{\psi}\gamma^i(\partial_i\gamma^0\psi) \\ &= -iG_i\gamma^5(\partial_i\bar{\psi})\psi + iG_i\gamma^5\bar{\psi}(\partial_i\psi). \end{aligned} \tag{A.28}$$

We see that when $\mu = i$, $F_5$is P odd.

Under charge conjugation,

$$\begin{aligned} F_{5C} &= -iG_\mu\gamma^5\left(\partial_\mu'\bar{\psi}'\right)\psi' + iG_\mu\gamma^5\bar{\psi}'(\partial_\mu{}'\psi') \\ &= -iG_\mu\gamma^5\left(\partial_\mu\bar{\psi}\gamma^0\gamma^2\right)\gamma^0\gamma^2\psi + iG_\mu\gamma^5\bar{\psi}\gamma^0\gamma^2(\partial_\mu\gamma^0\gamma^2\psi) \\ &= -iG_\mu\gamma^5\left(\partial_\mu\bar{\psi}\right)\psi + iG_\mu\gamma^5\bar{\psi}(\partial_\mu\psi). \end{aligned} \tag{A.29}$$

$F_5$is C odd.

Under time reversal,

When $\mu = 0$:

$$\begin{aligned} F_{5T} &= iG_0\gamma^5(\partial_0\bar{\psi}\gamma^1\gamma^3)\gamma^1\gamma^3\psi - iG_0\gamma^5\bar{\psi}\gamma^1\gamma^3(\partial_0\gamma^1\gamma^3\psi) \\ &= -iG_0\gamma^5(\partial_0\bar{\psi})\psi + iG_0\gamma^5\bar{\psi}(\partial_0\psi). \end{aligned} \tag{A.30}$$

We see that when $\mu = 0$, F_5 is T odd.

When $\mu = i$:

$$\begin{aligned} F_{5T} &= -iG_i\gamma^5(\partial_i\bar{\psi}\gamma^1\gamma^3)\gamma^1\gamma^3\psi + iG_i\gamma^i\bar{\psi}\gamma^1\gamma^3(\partial_i\gamma^1\gamma^3\psi) \\ &= iG_i\gamma^5(\partial_i\bar{\psi})\psi - iG_i\gamma^5\bar{\psi}(\partial_i\psi). \end{aligned} \tag{A.31}$$

We see that when $\mu = i$, $F_5$is T even.

Since we want CP odd, the final form of F_5 is given as such

$$F_5 = iG_0\gamma^5(\partial_0\bar{\psi})\psi - iG_0\gamma^5\bar{\psi}(\partial_0\psi). \tag{A.32}$$

Appendix B: Derivation of Other MDRs

For $\mathcal{L}_1$:

$$\mathcal{L}_1 = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi \ + A_\mu \bar{\psi}\gamma^\mu \psi. \tag{B.1}$$

$$\frac{\partial \mathcal{L}_1}{\partial \bar{\psi}} = i\gamma^\mu \partial_\mu \psi - m\psi + A_\mu \gamma^\mu \psi. \tag{B.2}$$

$$\frac{\partial \mathcal{L}_1}{\partial\left(\partial_\mu \bar{\psi}\right)} = 0 \Rightarrow \partial_\mu \left(\frac{\partial \mathcal{L}_1}{\partial\left(\partial_\mu \bar{\psi}\right)}\right) = 0. \tag{B.3}$$

Equating Equations (B.2) and (B.3),

$$\begin{gathered} i\gamma^\mu \partial_\mu \psi - m\psi + A_\mu \gamma^\mu \psi = 0 \\ \gamma^\mu k_\mu \psi - m\psi + A_\mu \gamma^\mu \psi = 0 \\ [\gamma^\mu (k_\mu + A_\mu) - m]\psi = 0 \\ (k_\mu + A_\mu)^2 - m^2 = 0 \\ {k_\mu}^2 + A_\mu^2 + 2k_\mu A_\mu - m^2 = 0 \\ \Rightarrow \boldsymbol{k}^2 = m^2 - \boldsymbol{A}^2 - 2\boldsymbol{k}.\boldsymbol{A}. \end{gathered} \tag{B.4}$$

For $\mathcal{L}_2$:

$$\mathcal{L}_2 = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi + B_\mu \bar{\psi}\gamma^\mu \gamma^5 \psi. \tag{B.5}$$

$$\frac{\partial \mathcal{L}_2}{\partial \bar{\psi}} = i\gamma^\mu \partial_\mu \psi - m\psi + B_\mu \gamma^\mu \gamma^5 \psi. \tag{B.6}$$

$$\frac{\partial \mathcal{L}_2}{\partial\left(\partial_\mu \bar{\psi}\right)} = 0 \Rightarrow \partial_\mu \left(\frac{\partial \mathcal{L}_2}{\partial\left(\partial_\mu \bar{\psi}\right)}\right) = 0. \tag{B.7}$$

Equating Equations (B.6) and (B.7),

$$(\gamma^\mu k_\mu + B_\mu \gamma^\mu \gamma^5 - m)\psi = 0. \tag{B.8}$$

After expanding Equation (B.8) and multiplying γ^0 from the left,

$$(\gamma^0 E + B_0\gamma^0\gamma^5)\psi = \left(-\alpha^i k_i - B_i\alpha^i\gamma^5 + m\gamma^0\right)\psi = 0. \tag{B.9}$$

where $\alpha^i = \gamma^0\gamma^i$

Squaring both sides of the equation, we get:

$$(E^2 + B_0^2 + 2EB_0\gamma^5)\psi = \left(k_i^2 + B_i^2 + m^2 + 2B_ik_i\gamma^5 - 2mB_i\gamma^i\gamma^5\right)\psi. \tag{B.10}$$

Since $\psi = \begin{pmatrix}\psi_a\\ \psi_b\end{pmatrix}$, $\gamma^5 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and $\gamma^i = \begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix}$, where σ^i are the Pauli matrices, Equation (B.10) becomes

$$(E^2 + B_0^2)\begin{pmatrix}\psi_a\\ \psi_b\end{pmatrix} + 2EB_0\begin{pmatrix}\psi_b\\ \psi_a\end{pmatrix} = \left(k_i^2 + B_i^2 + m^2\right)\begin{pmatrix}\psi_a\\ \psi_b\end{pmatrix} + 2B_ik_i\begin{pmatrix}\psi_b\\ \psi_a\end{pmatrix} - 2B_im\begin{pmatrix}\sigma^i\psi_a\\ -\sigma^i\psi_b\end{pmatrix}. \tag{B.11}$$

This leads to two simultaneous equations

$$\begin{gathered}(E^2 + B_0^2)\psi_a + 2EB_0\psi_b = \left(k_i^2 + B_i^2 + m^2\right)\psi_a + 2B_ik_i\psi_b - 2B_im\sigma^i\psi_a\\ (E^2 + B_0^2 - k_i^2 - B_i^2 - m^2 + 2B_im\sigma^i)\psi_a = (2B_ik_i - 2EB_0)\psi_b\\ \psi_a = \frac{2B_ik_i - 2EB_0}{E^2 + B_0^2 - k_i^2 - B_i^2 - m^2 + 2B_im\sigma^i}\psi_b.\end{gathered} \tag{B.12}$$

$$\begin{gathered}(E^2 + B_0^2)\psi_b + 2EB_0\psi_a = \left(k_i^2 + B_i^2 + m^2\right)\psi_b + 2B_ik_i\psi_a + 2B_im\sigma^i\psi_b\\ (2EB_0 - 2B_ik_i)\psi_a = (k_i^2 + B_i^2 + m^2 + 2B_im\sigma^i - E^2 - B_0^2)\psi_b\\ \psi_a = \frac{k_i^2 + B_i^2 + m^2 + 2B_im\sigma^i - E^2 - B_0^2}{2EB_0 - 2B_ik_i}\psi_b.\end{gathered} \tag{B.13}$$

Equating Equations (B.12) and (B.13), we will get the MDR,

$$(2k_iB_i - 2EB_0)^2 = (E^2 + B_0^2 - k_i^2 - B_i^2 - m^2 - 2B_im\sigma^i)(E^2 + B_0^2 - k_i^2 - B_i^2 - m^2 + 2B_im\sigma^i. \tag{B.14}$$

In order to have CP odd for F_2, $\mu = 0$, thus we can ignore the terms with $\mu = i$.

Eventually, we get the MDR for F_2

$$4E^2 B_0^2 = (\boldsymbol{k}^2 + B_0^2 - m^2)(\boldsymbol{k}^2 + B_0^2 - m^2)$$
$$4E^2 B_0^2 = \boldsymbol{k}^4 + B_0^4 + m^4 - 2\boldsymbol{k}^2 m^2 + 2\boldsymbol{k}^2 B_0^2 - 2m^2 B_0^2$$
$$\Rightarrow \boldsymbol{k}^2(\boldsymbol{k}^2 - 2m^2 + 2B_0^2) = 4E^2 B_0^2 + 2m^2 B_0^2 - B_0^4 - m^4. \tag{B.15}$$

For $\mathcal{L}_4$:

$$\mathcal{L}_4 = \bar{\psi}\big(i\gamma^\mu \partial_\mu - m\big)\psi + iD_\mu \gamma^\mu \big(\partial_\mu \bar{\psi}\big)\psi - iD_\mu \gamma^\mu \bar{\psi}(\partial_\mu \psi). \tag{B.16}$$

$$\frac{\partial \mathcal{L}_4}{\partial \bar{\psi}} = i\gamma^\mu \partial_\mu \psi - m\psi - iD_\mu \gamma^\mu (\partial_\mu \psi). \tag{B.17}$$

$$\frac{\partial \mathcal{L}_4}{\partial\big(\partial_\mu \bar{\psi}\big)} = iD_\mu \gamma^\mu \psi \Rightarrow \partial_\mu \left(\frac{\partial \mathcal{L}_4}{\partial\big(\partial_\mu \bar{\psi}\big)}\right) = iD_\mu \gamma^\mu \big(\partial_\mu \psi\big). \tag{B.18}$$

Equating Equations (B.17) and (B.18),

$$i\gamma^\mu \partial_\mu \psi - m\psi - iD_\mu \gamma^\mu \big(\partial_\mu \psi\big) = iD_\mu \gamma^\mu (\partial_\mu \psi)$$
$$\gamma^\mu k_\mu \psi - m\psi - 2iD_\mu \gamma^\mu \big(\partial_\mu \psi\big) = 0$$
$$\gamma^\mu k_\mu \psi - m\psi - 2D_\mu m\psi = 0$$
$$\gamma^\mu k_\mu \psi - \big(m + 2D_\mu m\big)\psi = 0$$
$$k_\mu^2 - \big(m + 2D_\mu m\big)^2 = 0$$
$$k_\mu^2 - (m^2 + 4D_\mu^2 m^2 + 4D_\mu m^2) = 0. \tag{B.19}$$

Since we require $\mu = 0$ to have CP odd for F_4, the final form of the MDR is,

$$\boldsymbol{k}^2 = m^2 + 4D_0^2 m^2 + 4D_0 m^2. \tag{B.20}$$

For $\mathcal{L}_5$:

$$\mathcal{L}_5 = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \ + iG_\mu\gamma^5(\partial_\mu\bar{\psi})\psi - iG_\mu\gamma^5\bar{\psi}(\partial_\mu\psi). \tag{B.21}$$

$$\frac{\partial\mathcal{L}_5}{\partial\bar{\psi}} = i\gamma^\mu\partial_\mu\psi - m\psi - iG_\mu\gamma^5(\partial_\mu\psi). \tag{B.22}$$

$$\frac{\partial\mathcal{L}_5}{\partial(\partial_\mu\bar{\psi})} = iG_\mu\gamma^5\psi \Rightarrow \partial_\mu\left(\frac{\partial\mathcal{L}_5}{\partial(\partial_\mu\bar{\psi})}\right) = iG_\mu\gamma^5(\partial_\mu\psi). \tag{B.23}$$

Equating Equations (B.22) and (B.23),

$$\begin{gathered} i\gamma^\mu\partial_\mu\psi - m\psi - iG_\mu\gamma^5(\partial_\mu\psi) = iG_\mu\gamma^5(\partial_\mu\psi) \\ \gamma^\mu k_\mu\psi - m\psi - 2iG_\mu\gamma^5(\partial_\mu\psi) = 0 \\ \gamma^\mu k_\mu\psi - m\psi - 2G_\mu m\gamma^5\gamma^\mu\psi = 0. \end{gathered} \tag{B.24}$$

After expanding Equation (B.24) and multiplying γ^0 from the left,

$$(E + 2mG_0\gamma^5)\psi = (\gamma^0 m - \alpha^i k_i - 2mG_i\gamma^5\alpha^i)\psi. \tag{B.25}$$

Squaring both sides of the equation, we get:

$$(E^2 + 4m^2G_0^2 + 4mEG_0\gamma^5)\psi = (m^2 - k_i^2 + 4m^2G_i\gamma^5\gamma^i + 4m^2G_i^2)\psi. \tag{B.26}$$

Since $\psi = \begin{pmatrix}\psi_a \\ \psi_b\end{pmatrix}$, $\gamma^5 = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}$ and $\gamma^i = \begin{pmatrix}\sigma^i & 0 \\ 0 & -\sigma^i\end{pmatrix}$, where σ^i are the Pauli matrices, Equation (B.26) becomes

$$(E^2 + 4m^2G_0^2)\begin{pmatrix}\psi_a \\ \psi_b\end{pmatrix} + 4mEG_0\begin{pmatrix}\psi_b \\ \psi_a\end{pmatrix} = (m^2 - k_i^2 + 4m^2G_i^2)\begin{pmatrix}\psi_a \\ \psi_b\end{pmatrix} + 4m^2G_i\begin{pmatrix}\sigma^i\psi_a \\ -\sigma^i\psi_b\end{pmatrix}. \tag{B.27}$$

This leads to two simultaneous equations

$$(E^2 + 4m^2G_0^2)\psi_a + 4mEG_0\psi_b = \left(m^2 - k_i^2 + 4m^2G_i^2\right)\psi_a - 4m^2G_i\sigma^i\psi_a$$
$$\psi_b = \frac{m^2 - k_i^2 + 4m^2G_i^2 - 4m^2G_im\sigma^i - E^2 - 4m^2G_0^2}{4mEG_0}\psi_a. \tag{B.28}$$

$$(E^2 + 4m^2G_0^2)\psi_b + 4mEG_0\psi_a = \left(m^2 - k_i^2 + 4m^2G_i^2\right)\psi_b + 4m^2G_i\sigma^i\psi_b$$
$$\psi_a = \frac{m^2 - k_i^2 + 4m^2G_i^2 + 4m^2G_im\sigma^i - E^2 - 4m^2G_0^2}{4mEG_0}\psi_b. \tag{B.29}$$

Substituting Equation (B.28) into Equation (B.29),

$$\psi_a = \frac{(m^2 - k_i^2 - E^2 - 4m^2G_0^2 + 4m^2G_i^2 + 4m^2G_i\sigma^i)(m^2 - k_i^2 - E^2 - 4m^2G_0^2 + 4m^2G_i^2 - 4m^2G_i\sigma^i)}{16m^2E_0^2G_0^2}\psi_a. \tag{B.30}$$

Let $a = m^2 - k_i^2 - E^2 - 4m^2G_0^2 + 4m^2G_i^2$ and $b = 4m^2G_i\sigma^i$, equation B.30 becomes,

$$\psi_a = \frac{a^2 - b^2}{16m^2E_0^2G_0^2}\psi_a$$
$$16m^2E_0^2G_0^2\psi_a = (a^2 - b^2)\psi_a. \tag{B.31}$$

With

$$a^2 = m^4 + \boldsymbol{k}^4 - 2m^2k^2 - 8m^4G_0^2 + 8m^4G_i^2 + 8m^2\boldsymbol{k}^2G_0^2 - 8m^2\boldsymbol{k}^2G_i^2 + 16m^4\boldsymbol{G}^4 - 32m^4G_0^2G_i^2. \tag{B.32}$$

Substituting back into Equation (B.31), we get the MDR

$$16m^2E_0^2G_0^2 = m^2 + \frac{\boldsymbol{k}^4}{m^2} - 2\boldsymbol{k}^2 - 8m^2\left(G_0^2 - G_i^2\right) + 8\boldsymbol{k}^2\left(G_0^2 - G_i^2\right) + 16m^2\boldsymbol{G}^4 - 32m^2G_0^2G_i^2$$
$$\boldsymbol{k}^2 = \frac{m^2[16m^2E_0^2G_0^2 - m^2 + 8m^2\left(G_0^2 - G_i^2\right) - 16m^2\boldsymbol{G}^4 + 32m^2G_0^2G_i^2]}{\boldsymbol{k}^2 - 2m^2 + 8m^2(G_0^2 - G_i^2)}. \tag{B.33}$$

Since we require $\mu = 0$ to have CP odd for F_5, the final form of the MDR is,

$$\boldsymbol{k}^2 = \frac{m^2[16m^2E_0^2G_0^2 - m^2 + 8m^2G_0^2 - 16m^2G_0^4]}{\boldsymbol{k}^2 - 2m^2 + 8m^2G_0^2}. \tag{B.34}$$

Appendix C: PNMS Matrix

The PNMS matrix is given as below:

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix} \begin{bmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{CP}} \\ 0 & 1 & 0 \\ -s_{13}e^{-i\delta_{CP}} & 0 & c_{13} \end{bmatrix} \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{12}c_{23} \end{pmatrix}$$

where $c_{ij} = \cos\theta_{ij}$and $s_{ij} = \sin\theta_{ij}$; θ_{ij} are the mixing angles and δ is the CP-violating phase.

Appendix D: Expansion of *Real* Part of Equation (5.20)

$$\boldsymbol{Re}\left(U_{\mu 1}U_{e1}^{*}U_{\mu 2}^{*}U_{e2}\right) = \boldsymbol{Re}[c_{12}c_{13}\left(-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta}\right)s_{12}c_{13}(c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta})$$

$$= \boldsymbol{Re}[c_{12}s_{12}c_{13}^{2}(s_{12}^{2}c_{23}s_{23}s_{13}e^{-i\varphi} - s_{12}c_{12}c_{23}^{2} + c_{12}s_{12}s_{13}s_{23}^{2}$$

$$-c_{12}^{2}c_{23}s_{23}s_{13}e^{i\varphi}]$$

$$= c_{12}s_{12}c_{13}^{2}(s_{12}^{2}c_{23}s_{23}s_{13}c_{\varphi} - s_{12}c_{12}c_{23}^{2} + c_{12}s_{12}s_{13}s_{23}^{2} - c_{12}^{2}c_{23}s_{23}s_{13}c_{\varphi}).$$

$$\boldsymbol{Re}\left(U_{\mu 1}U_{e1}^{*}U_{\mu 3}^{*}U_{e3}\right) = c_{12}s_{23}c_{13}^{2}s_{13}c_{\varphi}(-s_{12}c_{23} - c_{12}s_{23}s_{13}c_{\varphi}).$$

$$\boldsymbol{Re}\left(U_{\mu 2}U_{e2}^{*}U_{\mu 3}^{*}U_{e3}\right) = s_{12}s_{23}c_{13}^{2}s_{13}c_{\varphi}(c_{12}c_{23} - s_{12}s_{13}s_{23}c_{\varphi}).$$

Appendix E: Neutrino Oscillation Probabilities for Other MDRs

For $\mathcal{L}_1$:

The MDR is given as

$$\boldsymbol{k}^2 = m^2 - \boldsymbol{A}^2 - 2\boldsymbol{k}.\boldsymbol{A}\ . \tag{E.1}$$

Expanding k,

$$\begin{aligned} E^2 - {p_i}^2 &= m^2 - \boldsymbol{A}^2 - 2\boldsymbol{k}.\boldsymbol{A} \\ {p_i}^2 &= E^2 - m^2 + \boldsymbol{A}^2 + 2(E.A_0 + p.A_i) \\ \Rightarrow {p_i}^2 &= E^2 - m^2 + {A_0}^2 + 2E.A_0 \end{aligned} \tag{E.2}$$

since $\mu = 0$ for F_1.

Assuming $A_0$is small, we can neglect higher order terms and doing a Taylor expansion, we eventually get

$$\begin{aligned} p_i &= \sqrt{E^2 - {m_i}^2 + {A_0}_i^{\ 2} + 2E.{A_0}_i} \\ &= E\left(1 - \frac{{m_i}^2}{E^2} + \frac{{A_0}_i^{\ 2}}{E^2} + \frac{2{A_0}_i}{E}\right)^{\frac{1}{2}} \\ &\approx E\left[1 - \frac{1}{2}\left(\frac{{m_i}^2}{E^2} - \frac{2{A_0}_i}{E}\right)\right]. \end{aligned} \tag{E.3}$$

Equation (5.15) becomes

$$\phi_i = \left(\frac{{m_i}^2}{2E^2} - \frac{{A_0}_i}{E}\right)L. \tag{E.4}$$

The oscillation probability in the case of $\mathcal{L}_1$ is

$$\begin{aligned}\mathrm{P}(\nu_\mu \to \nu_e) = -4\boldsymbol{Re}\{U_{\mu1}U_{e1}^*U_{\mu2}^*U_{e2}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{21}^2}{2E} - \Delta A_{0_{21}}\right)\right] \\ +U_{\mu1}U_{e1}^*U_{\mu3}^*U_{e3}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{31}^2}{2E} - \Delta A_{0_{31}}\right)\right] \\ +U_{\mu2}U_{e2}^*U_{\mu3}^*U_{e3}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{32}^2}{2E} - \Delta A_{0_{32}}\right)\right]\}.\end{aligned} \tag{E.5}$$

For $\boldsymbol{\mathcal{L}_2}$:

The MDR is given as

$$\boldsymbol{k}^2(\boldsymbol{k}^2 - 2m^2 + 2B_0^2) = 4E^2B_0^2 + 2m^2B_0^2 - B_0^4 - m^4. \tag{E.6}$$

Expanding k,

$$\begin{aligned}(E^2 - p_i^2)(E^2 - p_i^2 - 2m^2 + 2B_0^2) = 4E^2B_0^2 + 2m^2B_0^2 - B_0^4 - m^4 \\ (E^2 - p_i^2) = m^2 - B_0^2 + 2EB_0.\end{aligned} \tag{E.7}$$

Assuming $B_0$is small, we can neglect higher order terms and doing a Taylor expansion, we eventually get

$$\begin{aligned}p_i &= \sqrt{E^2 - {m_i}^2 + {B_{0_i}}^2 + 2E.B_{0_i}} \\ &= E\left(1 - \frac{{m_i}^2}{E^2} + \frac{{B_{0_i}}^2}{E^2} + \frac{2B_{0_i}}{E}\right)^{\frac{1}{2}} \\ &\approx E\left[1 - \frac{{m_i}^2}{2E^2} + \frac{B_{0_i}}{E}\right].\end{aligned} \tag{E.8}$$

Equation (5.15) becomes

$$\phi_i = \left(\frac{{m_i}^2}{2E^2} - \frac{B_{0_i}}{E}\right)L. \tag{E.9}$$

The oscillation probability in the case of $\mathcal{L}_2$ is

$$\begin{aligned}
\mathrm{P}(\nu_\mu \to \nu_e) = -4\boldsymbol{Re}\{&U_{\mu1}U_{e1}^*U_{\mu2}^*U_{e2}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{21}^2}{2E} - \Delta B_{0_{21}}\right)\right] \\
&+U_{\mu1}U_{e1}^*U_{\mu3}^*U_{e3}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{31}^2}{2E} - \Delta B_{0_{31}}\right)\right] \\
&+U_{\mu2}U_{e2}^*U_{\mu3}^*U_{e3}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{32}^2}{2E} - \Delta B_{0_{32}}\right)\right]\}. \qquad (E.10)
\end{aligned}$$

For $\mathcal{L}_4$:

The MDR is given as

$$\boldsymbol{k}^2 = m^2 + 4D_0^2 m^2 + 4D_0 m^2. \qquad (E.11)$$

Expanding k,

$$E^2 - p_i^2 = m^2 + 4D_0^2 m^2 + 4D_0 m^2. \qquad (E.12)$$

Assuming $D_0$is small, we can neglect higher order terms and doing a Taylor expansion, we eventually get

$$\begin{aligned}
p_i &= \sqrt{E^2 - {m_i}^2(1 + 4{D_0}_i{}^2 + 4{D_0}_i)} \\
&= E\left(1 - \frac{{m_i}^2}{E^2}(1 + 4{D_0}_i{}^2 + 4{D_0}_i)\right)^{\frac{1}{2}} \\
&\approx E\left[1 - \frac{{m_i}^2}{2E^2}(1 + 4{D_0}_i)\right]. \qquad (E.13)
\end{aligned}$$

Equation (5.15) becomes

$$\phi_i = \frac{{m_i}^2}{2E^2}(1 + {D_0}_i)L. \qquad (E.14)$$

The oscillation probability in the case of $\mathcal{L}_2$ is

$$\begin{aligned}
\mathrm{P}\left(\nu_\mu \to \nu_e\right) = -4\boldsymbol{Re}\{&U_{\mu1}U_{e1}^*U_{\mu2}^*U_{e2}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{21}^2}{2E}\left(1+4\Delta D_{0\,21}\right)\right)\right] \\
&+U_{\mu1}U_{e1}^*U_{\mu3}^*U_{e3}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{31}^2}{2E}\left(1+4\Delta D_{0\,21}\right)\right)\right] \\
&+U_{\mu2}U_{e2}^*U_{\mu3}^*U_{e3}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{32}^2}{2E}\left(1+4\Delta D_{0\,21}\right)\right)\right]\}. \qquad \text{(E.15)}
\end{aligned}$$

For $\mathcal{L}_5$:

The MDR is given as

$$\boldsymbol{k}^2 = \frac{m^2[16m^2E_0^2G_0^2 - m^2 + 8m^2G_0^2 - 16m^2G_0^4]}{\boldsymbol{k}^2 - 2m^2 + 8m^2G_0^2}. \qquad \text{(E.16)}$$

Expanding k,

$$(E^2 - p_i^2)(E^2 - p_i^2 - 2m^2 + 8m^2G_0^2) = m^2[16m^2E_0^2G_0^2 - m^2 + 8m^2G_0^2 - 16m^2G_0^4]$$

$$E^2 - p_i^2 = m^2 - 4m^2G_0^2 - 4mEG_0. \qquad \text{(E.17)}$$

Assuming $G_0$is small, we can neglect higher order terms and doing a Taylor expansion, we eventually get

$$\begin{aligned}
p_i &= \sqrt{E^2 - {m_i}^2 + 4{m_i}^2{G_0}_i^{\,2} + 4m_iE{G_0}_i)} \\
&= E\left(1 - \frac{{m_i}^2}{E^2} + \frac{4{m_i}^2{G_0}_i^{\,2}}{E^2} + \frac{4m_i{G_0}_i}{E})\right)^{\frac{1}{2}} \\
&\approx E\left[1 - \frac{1}{2}\left(\frac{{m_i}^2}{E^2} + \frac{4m_i{G_0}_i}{E}\right)\right]. \qquad \text{(E.18)}
\end{aligned}$$

Equation (5.15) becomes

$$\phi_i = \left(\frac{{m_i}^2}{2E^2} - 2m_i{G_0}_i\right)L. \qquad \text{(E.19)}$$

The oscillation probability in the case of $\mathcal{L}_2$ is

$$\mathrm{P}\left(\nu_\mu \to \nu_e\right) = -4\boldsymbol{Re}\{U_{\mu1}U_{e1}^*U_{\mu2}^*U_{e2}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{21}^2}{2E} - 2\Delta m_{21}\Delta G_{0_{21}}\right)\right]$$

$$+U_{\mu1}U_{e1}^*U_{\mu3}^*U_{e3}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{31}^2}{2E} - 2\Delta m_{31}\Delta G_{0_{31}}\right)\right]$$

$$+U_{\mu2}U_{e2}^*U_{\mu3}^*U_{e3}\sin^2\left[\frac{L}{2}\left(\frac{\Delta m_{32}^2}{2E} - 2\Delta m_{32}\Delta G_{0_{32}}\right)\right]\}. \qquad \text{(E.20)}$$