ROTATING DUST MODELS IN GENERAL RELATIVITY

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Abstract

It is widely accepted that a dark matter halo should explain the flat rotation curves problem of galaxies. However, two sets of authors, Cooperstock & Tieu [8] and Balasin & Grumiller [1], have claimed that general relativistic models of rotating galaxies could reproduce the correct rotation curves without introducing dark matter. Inspired by these works, this project gives a short review on the general methods of solving Einstein's equation for an axially-symmetric and stationary system of dust and analyzes six of such solutions in terms of their physical properties, including those of Cooperstock & Tieu and Balasin & Grumiller and one that is original. We did not identify a perfect solution to solve the rotation curves problem, and future work can be done by looking at more solutions or formulating an analytic condition on the solutions.

Note A large proportion of this project is review on work that has been done, but there is also quite some original work, including the second method of generating K in Eq. 3.14, the formula identifying an angular velocity in different frames (Eq. 3.46), proof that the density expression in Cooperstock & Tieu's solution vanishes at infinity (later parts of section 3.5) and the construction and primary analysis of the "Rod" solution (section 3.6). It might happen that these results have been obtained in other scientific research work, but I am not aware of them at the point of submitting this thesis.

Contents

List of Figures

CHAPTER 1

Motivation: Dark Matter & Rotation Curves

1.1 Dark Matter: A Historical Perspective

Before the late 30s of last century, astronomers usually referred to the famous *mass-to-luminosity relation* to estimate the mass of a distant star based on photometric measurements:

$$\frac{L}{L_0} = \left(\frac{M}{M_0}\right)^a \tag{1.1}$$

Here L and M refers to the luminosity and mass of the distant star, L_0 and M_0 to those of a known star (typically, the sun $\odot$). a is a parameter that ranges from 1 to 6, dependent on the luminosity of the star. This empirical relation could consequently give the mass of a distant galaxy or even a cluster, given that one knows how many stars or galaxies are there. However, when Zwicky [21] studied the *Coma Cluster* using the virial theorem,

$$2\langle T\rangle + \langle U\rangle = 0 \tag{1.2}$$

he arrived at an estimated mass of the order of $10^{14}M_\odot$. This number is almost 10^3 times of that estimated using the M/L relation, leading him to suspect that there is a large amount of mass in the cluster that does not emit light. Meanwhile, Dutch astronomer Oort [12] published his paper on the velocity of stars near the Milky Way center, which also supports the suspicion that a large proportion of the mass in our galaxy is not visible. The term, *dark matter*, adequately describes this kind of material that is dominant and yet not luminous.

In the 1970s, Rubin [13] provided another strong supporting argument for dark matter in the study of *rotation curves.* She observed that when the distance (r) between visible stars and the galactic center is large enough, the rotational speed (v) tends to be constant (Fig. 1.1).

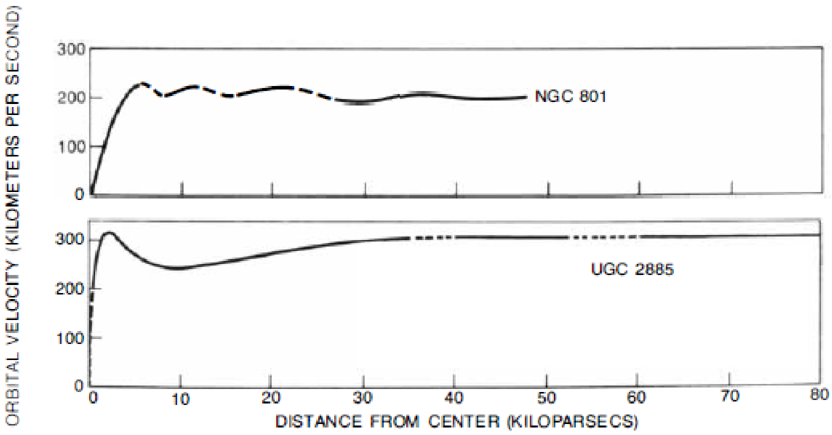

Figure 1.1: Rotation curves of two spiral galaxies [13]

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This phenomenon contradicts the photometric observation that most of the galactic matter is concentrated near the central bulge. To see this, we assume spherical symmetry of the system. With Newtonian physics, it is readily seen that,

$$v^2 = G\frac{M(r)}{r} \tag{1.3}$$

where G is the gravitational constant, $M(r)$ the mass enclosed in radius r. Since matter is concentrated near the galactic center, we could expect a Keplerian behavior for $v(r)$, i.e., $v \sim \frac{1}{\sqrt{r}}$. The flat plateaus in rotation curves are certainly not Keplerian; it suggests that there is a huge amount of "missing" mass that contributes gravitationally.

More evidence of dark matter came into the picture when the *Big Bang theory* was proposed, and some attribute the large fluctuation of matter during the recombination epoch to the existence of dark mater [11]; gravitational lensing, on the other hand, has indirectly observed dark matter [4].

1.2 Attempts to Explain the Missing Mass Puzzle

If dark matter does exist, the first question that one asks is what it is. So far what dark matter consists of is still an unsolved mystery. We know certain properties, e.g., that it should not interact electromagnetically but gravitationally; that it should be massive and abundant. The first property leads to apparent difficulty in detection, and so far there is no direct detection of dark matter.

There is a wide range of popular candidates and their related theories. The standard model neutrino, for example, is known and well studied, but a calculation of its abundance indicates that there are not enough neutrinos to account for so large a discrepancy in the cosmological models [5].

More exotic candidates include axions and various supersymmetric partners of the particles we know. However, since the theories behind them are still in construction, these candidates either pose a significant uncertainty bar or arouse controversy in their existence [5].

On the other side of the path, some try to modify Newtonian dynamics to explain the missing mass problems. *MOdified Newtonian Dynamcis* (MOND), for example, argues that gravity would take a different form at different length scales or acceleration [14], and has succeeded in producing the desired rotation curves. An insurmountable difficulty, however, is that the scale at which gravity should start to deviate from Newtonian is different for galaxies of different luminosities. This seems to be a more empirical approach that hints at a more involved theory behind.

An apparent candidate of modified Newtonian gravity, that has a solid theoretical background, is general relativity (GR). It had not been a popular choice to explain rotation curves until recently: a few published papers claim that GR could answer this problem. Cooperstock & Tieu [8] and Balasin & Grumiller [1] published their work at about the same time, and in both papers a solution to Einstein's equation was found, producing satisfactory rotation curves (Fig. 1.2).

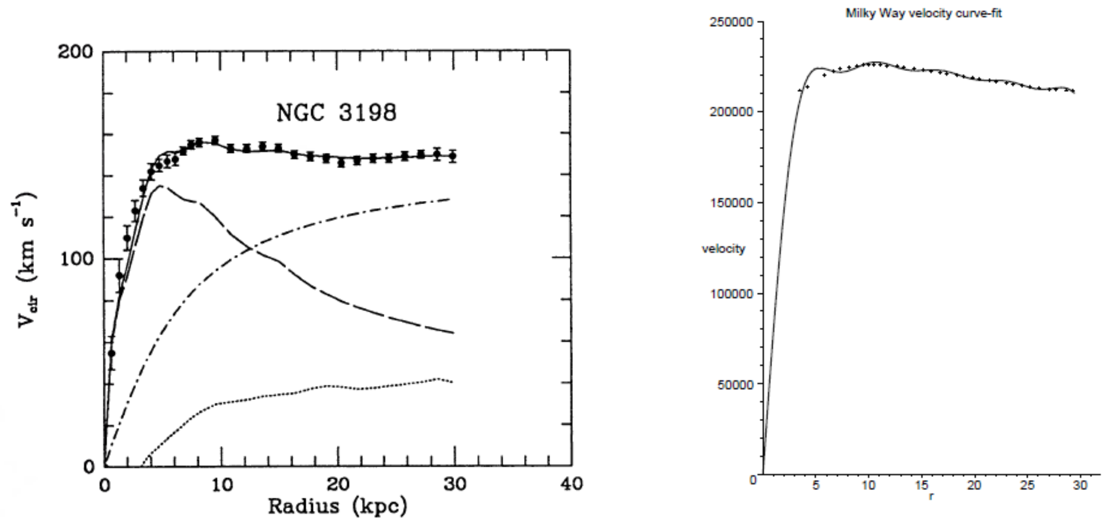

Figure 1.2: Left: fitting of the observed rotation curve with a dark-matter dominated model. Rotation induced from the individual components are also plotted: visible components (dashed), gas (dotted) and dark halo (dash-dot) [2]. More such fitting curves can be found in the paper [2].
Right: Fitting Cooperstock & Tieu's model [8] with data from the Milky Way. Units: m/s for velocity (vertical axis), Kpc for radius (horizontal axis). Balasin & Grumiller [1] obtained a similar result (Fig. 3.2).

One concern regarding this kind of GR approach is that it should not produce rotation curves very different from Newtonian gravity, given that the gravitational field is weak. Cooperstock & Tieu point out, in a later paper [7], that a global solution provided by GR has some intrinsic features that a linearised approach could not capture.

If these claims are indeed true, we might want to ask if GR could give rise to other dark matter phenomena as well. This project, inspired by these works, is dedicated to investigate these proposed solutions and their related topics. We start by looking at the general problem of rotating galaxies (Chapter 2), then proceed to rigid rotation solutions (Chapter 3), finally, remove the rigid rotation constraint (Chapter 4). A brief summary can be found at the end of the thesis (Chapter 5).

CHAPTER 2

Symmetries and Assumptions

The general problem of rotating galaxies can be simplified greatly with two assumptions:

- The system is axially symmetric and stationary
- The material inside the system can be approximated as dust

Given that galaxies are disk-like and the system is of astronomical scales, it is reasonable to assume that the system is rotationally invariant. On the other hand, we want a stable system that does not evolve in time. The second assumption is based on the observation that the density of a galaxy is generally low, which implies that most of the particles (or stars) should seldom interact with each other other than gravitationally.

We will discuss these two assumptions in more detail, but before that, we will try to give some comments on modelling a disk-like rotating galaxy using Newtonian methods.

2.1 Newtonian Methods

We have seen from Rubin's work [13] that the rotation curve deduced from luminous components is significantly different from that directly observed. However, it was in the first place questionable to impose spherical symmetry on the gravity models, as spiral galaxies are rather disk-like. Some work has been dedicated to the understanding of the system in a flattened axially-symmetric system. In these systems, a two-dimensional approach is used to calculate the gravitational potential,

$$\Phi(r') = 2\int_0^\pi d\theta \int_0^\infty dr \frac{r\rho(r)}{(r^2 + r'^2 - 2rr'\cos\theta)^{\frac{1}{2}}} \tag{2.1}$$

with $\Phi(r')$ the gravitational potential at r' and $\rho(r)$ the surface density of matter at r. By constructing proper ansats for $\rho(r)$, Mestel [10] could obtain curves of constant angular or tangential velocity similar to the observed rotation curves [1].

Nevertheless, this picture of a flattened axially-symmetric system is not realistic. As argued by Bonnor [6], no Newtonian solution can afford to be both axially-symmetric and finite along the z axis. The argument goes as the following:

Assume the velocity of dust particles $v(r)$ is represented as a three-vector of space in the cylindrical form. Denote the coordinates (r, z, φ) as (x^1, x^2, x^3),

$$u^i = (0, 0, \omega) \qquad u_j = g_{ij} u^i = (0, 0, \omega r^2) \tag{2.2}$$

the equation of motion gives,

$$u^i \nabla_i u_j = u^i \partial_i u_j - u^i \Gamma^k_{ij} u_k = -\partial_j \psi \tag{2.3}$$

where ψ is the gravitational field potential. This implies

$$-\partial_r \psi = -\omega^2 r \qquad \partial_z \psi = 0 \tag{2.4}$$

that ψ does not have a z dependence. The field has to extend infinitely along the axis of rotation.

Consequently, the density ρ that constructs the field via $\partial_i \partial^i \psi = 4\pi\rho$, can only be a function of r as well. In short:

$$\frac{d\psi}{dz} = 0 \qquad \frac{d\rho}{dz} = 0 \tag{2.5}$$

The model of a cylindrically symmetric galaxy fails the initial attempt to find a flattened system, regardless of the existence of dark matter. This argument is consistent with

[1] Specifically, a finite disk of uniform surface density produces constant angular velocity near the origin, but also a divergent gravitational acceleration near the edge of the disk. Alternatively, disks of density $\rho(r) \propto \frac{1}{r}$ give constant tangential velocity near the origin. [10] gives a comprehensive analysis.

our intuition. If the system is of finite length in the z direction, then the dust particles attract each other inevitably, resulting in an unstable system. Only if the system extends infinitely will this collapse be prevented.

Bonnor further notices that, in GR this conclusion will no longer be true, as spins of two massive bodies can provide a repulsive or attractive force along the z axis like dipoles [18]. Therefore, there is some interest in modelling the galaxy gravity with GR, in hope that this could give an alternative model of the rotating galaxies. The idea of an axially symmetric system, nevertheless, is a good starting point when one walks into the GR regime.

The following sections will be discussing the assumptions listed in the beginning of this chapter in the realm of GR.

2.2 Axially Symmetric and Stationary Metric

Start from Einstein's equation,

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi T^{\mu\nu} \tag{2.6}$$

The left hand side involves the Ricci Tensor, $R^{\mu\nu}$, the Ricci Scalar, R, and the metric $g^{\mu\nu}$. Since $R^{\mu\nu}$ and R are derived quantities from the metric $g^{\mu\nu}$, the problem reduces to finding a good metric that solves the equation. By imposing certain symmetry properties, the metric can be simplified to some general form. Assuming that the space-time is stationary and rotationally symmetric, we arrive at a compact form

$$ds^2 = f dt^2 - 2k dt d\varphi - l d\varphi^2 - e^{\mu}(dr^2 + dz^2) \tag{2.7}$$

where f, k, l and μ are functions of r and z only.

The first step in deriving this metric is to demand a metric that is invariant under the combined transformation of $\varphi \to -\varphi$ and $t \to -t$ but not under any flipping of individual coordinate directions, based on the assumption that the system is **rotating**. The coupling of dt and $d\varphi$ then remains while all the other off-diagonal terms involving dt or $d\varphi$ disappear. The coupling of dr and dz can be absorbed by a coordinate transformation [9]. In short, only one of the diagonal terms, $dtd\varphi$ is nonzero.

Secondly, by demanding **stationarity** and **axial symmetry**, the two killing vectors, one time-like and one space-like:

$$\boldsymbol{\zeta} = \frac{\partial}{\partial t} \qquad\qquad \boldsymbol{\eta} = \frac{\partial}{\partial \varphi} \tag{2.8}$$

apply to the space-time, so none of the functions in Eq. 2.7 are dependent on t or φ, as agrees with intuition.

One can then move on to talk about the motion of matter, which is taken to be a steady rotation along the $\hat{\boldsymbol{\varphi}}$ direction. If we denote the coordinates (t, r, z, φ) as (x^0, x^1, x^2, x^3), we can write down the four-velocity of the matter as:

$$u^1 = u^2 = 0 \qquad\qquad u^3 = \Omega u^0 \tag{2.9}$$

which can be explicitly written down given the metric (2.7)

$$u^\mu = (f - 2\Omega k - \Omega^2 l)^{-\frac{1}{2}}(1, 0, 0, \Omega) \tag{2.10}$$

2.3 Dust Approximation

The energy momentum tensor $T^{\mu\nu}$ describes the distribution of energy density ε and pressure per particle p for the gravitational field. For dust particles,

$$T^{\mu\nu} = mnu^\mu u^\nu \tag{2.11}$$

as the world lines of dust particles, by assumption, should not cross. Now the conservation of energy and matter links the two sides of Eq. 2.6,

$$\nabla_\nu T^{\mu\nu} = 0 \qquad \Rightarrow \qquad \nabla_\nu(nu^\mu u^\nu) = 0 \tag{2.12}$$

As in from conservation of matter, $\nabla_\nu(nu^\nu) = 0$, it is then clear that:

$$(u^\nu \nabla_\nu) u^\mu = \frac{du^\mu}{ds} + \Gamma^\mu_{\lambda\sigma} u^\lambda u^\sigma = 0 \tag{2.13}$$

which is reassuringly the geodesic equation.

Chapter 3

General Relativistic Dust Models: Rigid Rotation

Van Stockum (1937 [16]), Bonnor (1977 [6]), Cooperstock & Tieu (2008 [8]) and Balasin & Grumiller (2008 [1]) have all proposed similar methods to solve for a stationary and axially symmetric metric generated by pressure-less dust, and all under the assumption of **rigid rotation**, i.e., Ω in the four velocity of the dust is a constant.

It is not intuitive at the first sight why rigid rotation should be employed, as the rotation curve certainly demands an angular velocity that decreases after reaching the plateau. The point is, rigid rotation only assumes a constant coordinate angular velocity of the dust particles with respect to the coordinates, not putting any constraints on the part of the velocity induced by GR effects. In the following sections, especially section 3.3, a more detailed discussion on how to interpret this *angular velocity* will be presented.

Rigid rotation is not a necessary condition, but largely simplifies the problem. It turns out, if one follows the derivations below, all solutions differ only in the choice of one function (Eq. 3.12). Van Stockum's solution is a GR analog of the Newtonian infinite cylinder; Bonnor identified an asymptotically flat model but with a singularity at the origin; Cooperstock & Tieu and Balasin & Grumiller both came up with solutions that could fit the observational data rather well. This chapter is devoted to examining these solutions and their claims. In addition, some relevant issues (angular velocity and boundary conditions) will also be discussed.

3.1 General Approach

Here we introduce the general approach to solve Einstein's equation (2.6) using the axially symmetric stationary metric (2.7), following the approach adopted by Islam [9].

Multiplying $g_{\mu\nu}$ on both sides of Eq. 2.6, we can express R in terms of $-8\pi mn$, and therefore,

$$R_{\mu\nu} = 8\pi mn \left(u_\mu u_\nu - \frac{1}{2} g_{\mu\nu} \right) \tag{3.1}$$

First we observe the following equality implied from the metric (2.7) and the four velocity (2.10):

$$l \left(u_0 u_0 - \frac{1}{2} g_{00} \right) - 2k \left(u_0 u_3 - \frac{1}{2} g_{03} \right) - f \left(u_3 u_3 - \frac{1}{2} g_{33} \right) = 0 \tag{3.2}$$

which corresponds to

$$lR_{00} - 2kR_{03} - fR_{33} = 0 \tag{3.3}$$

on the left hand side of Eq. 3.1. Referring to the calculated quantities in Appendix (A.1) and the definition that $D^2 = fl + k^2$, one can arrive at the following equation

$$({\partial_r}^2 + {\partial_z}^2) D = 0 \tag{3.4}$$

Therefore, we can choose D to be r, as if a transformation of coordinates such that r satisfies $r^2 = fl + k^2$ is carried out. Any other choices of D can be recovered through an appropriate coordinate transformation [9](Section 4.2).

On the other hand, if one looks at $\mu = 1$ and 2 in the geodesic equation, Eq. 2.13, with the relevant Christoffel symbols (see Appendix A.1), it is obvious that

$$\frac{\partial f}{\partial r} - 2\Omega \frac{\partial k}{\partial r} - \Omega^2 \frac{\partial l}{\partial r} = 0 \tag{3.5}$$

$$\frac{\partial f}{\partial z} - 2\Omega \frac{\partial k}{\partial z} - \Omega^2 \frac{\partial l}{\partial z} = 0 \tag{3.6}$$

When Ω is a constant, $f - 2\Omega k - \Omega^2 l$ becomes a constant function, too (in the previous chapter, it was already shown that f, k and l are functions of r and z only). This prompts us to define a new set of coordinates, very essentially, in the co-moving frame, through the global transformation

$$\varphi \to \varphi + \Omega t \tag{3.7}$$

which implies that

$$f \to F \equiv f - 2\Omega k - \Omega^2 l \qquad k \to K \equiv k + \Omega l \qquad l \to L \equiv l \tag{3.8}$$

One can set $F = 1$ without loss of generality. Such a transformation results in the same form of the metric in Eq. 2.7 but with F, K and L:

$$ds^2 = F dt^2 - 2K dt d\varphi - L d\varphi^2 - e^{\mu}(dr^2 + dz^2) \tag{3.9}$$

Notice that this new frame is equivalent to the **comoving frame of the dust particles**, as can be seen from the way φ is transformed. The dust particles now have a different four-velocity

$$u^{\mu} = (1, 0, 0, 0) \tag{3.10}$$

at rest, as expected.

Secondly, we arrive at another relation implied by Einstein's equation, in the same vein as how we derived Eq. 3.3,

$$(k + \Omega l)R_{00} + (f + \Omega^2 l)R_{03} + \Omega(f - \Omega k)R_{33} = 0 \tag{3.11}$$

and expressed in the comoving frame,

$$K_{rr} + K_{zz} - \frac{K_r}{r} \equiv \Delta K = 0 \tag{3.12}$$

Eq. 3.12 will be the most important equation for this chapter. After choosing a function $K(r, z)$ that satisfies Eq. 3.12, $L(r, z)$ can be obtained from $FL + K^2 = r^2$ (it still holds after the transformation to the comoving frame [1]). Later on it will also be shown that $\mu(r,z)$ is determined by K through two partial differential equations. Eq. 3.12 looks very much like a Laplace equation in cylindrical coordinates, but differs by a sign in the first derivative term. Two ways of constructing this $K(r,z)$ are

$$K = rv_r \tag{3.13}$$

$$\text{or} \quad K_z = ru_r, \quad K_r = -ru_z \tag{3.14}$$

where $v(r,z)$ and $u(r,z)$ are harmonic functions, which means

$$v_{rr} + v_{zz} + \frac{1}{r}v_r = 0 \tag{3.15}$$

$$u_{rr} + u_{zz} + \frac{1}{r}u_r = 0 \tag{3.16}$$

When $K = rv_r$, we have

$$K_r = r(v_{rr} + \frac{1}{r}v_r) = -rv_{zz} \tag{3.17}$$

$$K_{rr} = -v_{zz} - rv_{zzr} \tag{3.18}$$

$$K_{zz} = (rv_{rz})_z = rv_{rzz} \tag{3.19}$$

which shows that

$$K_{rr} + K_{zz} - \frac{1}{r}K_r = -v_{zz} - rv_{zzr} + rv_{rzz} + v_{zz} = 0 \tag{3.20}$$

When $K_z = ru_r$, $K_r = -ru_z$,

$$K_{zz} = ru_{rz} \tag{3.21}$$

$$K_{rr} = -u_z - ru_{zr} \tag{3.22}$$

$$K_{rr} + K_{zz} - \frac{1}{r}K_r = ru_{rz} - u_z - ru_{zr} - \frac{1}{r}(-ru_z) = 0 \tag{3.23}$$

[1] $FL + K^2 = (f - 2\Omega k - \Omega^2 l)l + (k + \Omega l)^2 = fl - 2\Omega kl - \Omega^2 l^2 + k^2 + 2\Omega kl + \Omega^2 l^2 = fl + k^2$

It might not be obviously seen that u is necessarily harmonic, but the partial derivatives commute, so

$$K_{zr} - K_{rz} = u_r + ru_{rr} + ru_{zz} = 0 \tag{3.24}$$

Equivalently, $\frac{1}{r}u_r + u_{rr} + u_{zz} = 0$, and u is also harmonic. The first method is traditionally used to generate solutions for K (Van Stockum, Bonnor and Cooperstock & Tieu). The second method is an alternative method that we used to generate the "Rod" solution in section 3.6.

Next, looking at Eq. 3.1 with R_{11}, R_{22} and R_{12} for the relation between μ and K,

$$R_{11} = -\frac{1}{2}\mu_{rr} - \frac{1}{2}\mu_{zz} + \frac{1}{2}\frac{\mu_r}{r} + \frac{1}{2}\frac{{K_r}^2}{r^2} = 4\pi mne^{\mu} \tag{3.25}$$

$$R_{22} = -\frac{1}{2}\mu_{rr} - \frac{1}{2}\mu_{zz} - \frac{1}{2}\frac{\mu_r}{r} + \frac{1}{2}\frac{{K_z}^2}{r^2} = 4\pi mne^{\mu} \tag{3.26}$$

$$R_{12} = \frac{1}{2}\frac{\mu_z}{r} + \frac{1}{4}\frac{2K_rK_z}{r^2} = 0 \tag{3.27}$$

one finds the two partial differential equations of interest,

$$\mu_r = \frac{1}{2r}({K_z}^2 - {K_r}^2) \qquad \mu_z = -\frac{1}{r}K_rK_z \tag{3.28}$$

for the expression of μ. So far, all functions of the metric have been fixed in terms of a choice of $K(r, z)$ satisfying Eq. 3.12.

Finally, we ask for a quantity that gives us the physical insight into the system, which is mn, the mass density of the particles, related to the metric functions through

$$8\pi mn = \frac{{K_r}^2 + {K_z}^2}{r^2e^{\mu}} \tag{3.29}$$

which we could derive through

$$R_{33} = -\frac{1}{2}e^{-\mu}\left(\Delta L + \frac{L}{r^2}({K_r}^2 + {K_z}^2)\right) = 4\pi mn\frac{FL + 2K^2}{f} \tag{3.30}$$

and according to the choice of D and F

$$\Delta L = \Delta(\frac{r^2 - K^2}{F}) = -2(K_r^2 + K_z^2) \tag{3.31}$$

with $\Delta K = 0$ from Eq. 3.12. Therefore we can simplify Eq. 3.30 to be

$$R_{33} = \frac{1}{2}e^{-\mu}\frac{{K_r}^2 + {K_z}^2}{r^2}\left(2r^2 - L\right) = 4\pi mn\frac{FL + 2K^2}{f} \tag{3.32}$$

and because $F = 1$,

$$2r^2 - L = 2r^2 - FL = 2(FL + K^2) - FL = FL + 2K^2 \tag{3.33}$$

This completes a general method used to search for a solution to the rigid rotation problem.

3.2 Van Stockum's Solution

In the simplest case, if one demands that the solution to Eq. 3.12 is independent of z (which might not be physical), it immediately follows that K has the following form,

$$K = \alpha r^2 + \beta \tag{3.34}$$

Van Stockum chooses the specific form of K to be

$$K = \alpha r^2 \tag{3.35}$$

and the rest of the metric components follow,

$$L = r^2(1 - \alpha^2 r^2) \qquad \mu = -\alpha^2 r^2 \tag{3.36}$$

with a mass density of particles of

$$mn = \frac{\alpha^2 e^{\alpha^2 r^2}}{2\pi} \tag{3.37}$$

Since we are solving differential equations, boundary conditions are necessary for all solutions. In this case, integration constants are set to 0 without loss of generality. For example, there could be a constant added to the expression of μ in Eq. 3.36, but that could be interpreted as a scaling in the radial and axial directions without modifying the solution's essential properties (like the existence of a singularity surface, the shape of rotation curves and the divergence of mass density; more on this matter in the next section). Similarly, there might be an integration constant C added to K, but it is equivalent to the transformation of $t \to t + C\varphi$, as if another rigid rotation is superimposed onto this frame. In that case, it can be seen as a modification to Ω instead of the solution itself. Other constants, for example, α, can only be derived using some physical assumptions or observations, and it is left as a degree of freedom here.

Physical interpretation of this model suggests that it is not asymptotically flat; the mass distribution density goes up to infinity exponentially (and with a quadratic term). Furthermore, the solution clearly has the wrong symmetry. As discussed, we regard galaxies to be rather flat, but Van Stockum's solution extends infinitely along the z axis.

In addition to these undesired properties, the angular velocity of this solution is also not satisfactory for the modelling of galaxies (Eq. 3.47 in the next section). Under any circumstance, Van Stockum's solution alone cannot describe the physical properties of a rotating galaxy. Nevertheless, it is one of the earliest attempts to describe rotating galaxies, and will be a useful toy model for us to investigate some general properties.

3.3 Angular Velocity of the Dust

3.3.1 The Co-Moving Frame and the Locally Non-Rotating Frame

The transformation performed in Eq. 3.8 from f, k, l to F, K, L results in particles appearing to be **at rest**, as indicated by Eq. 3.10 (one can also calculate the Ω in this frame through the geodesic equation, and conclude that $\Omega = 0$ is indeed a solution. Calculations can be found in the Appendix A.2.1). On the other hand, as the metric (Eq. 3.9) is still not diagonal (for K is nonzero), in this particular frame space-time itself is rotating; the dust particles, therefore, are at rest with respect to the coordinates defined, but dragged around by the space-time.

This argument can be mathematically expressed via a transformation to a **locally non-rotating frame** [6]. Observers in this frame have the following metric,

$$ds^2 = F'dt^2 - L'd\varphi'^2 - e^{\mu}(dr^2 + dz^2) \tag{3.38}$$

which has no cross terms of $d\varphi'$ and dt. The angular velocity observed by observers in this nonrotating frame can be expressed as,

$$\omega' = \frac{d\varphi'}{dt} \tag{3.39}$$

We choose $\omega(r, z)$ in a local transformation

$$d\varphi' = d\varphi + \omega(r, z)dt \tag{3.40}$$

such that the metric is diagonalized, and it turns out that ω has to satisfy

$$\omega(r, z) = \frac{K}{L} \tag{3.41}$$

to complete the squares. It then can be easily seen from these two equations (3.40, 3.41) that, angular velocities observed in the locally nonrotating frame ω' relate to those observed in the comoving frame $\frac{d\varphi}{dt}$ via

$$\omega' = \frac{d\varphi}{dt} + \frac{K}{L} \tag{3.42}$$

and since the dust particles are at rest in the comoving frame,

$$\omega' = \frac{K}{L} \tag{3.43}$$

they rotate with respect to the locally nonrotating observers at the rate of $\frac{K}{L}$.

One might ask if the angular velocity Ω defined in the previous chapter (for example, in the four-velocity in Eq. 2.9) and also in the derivation of the general method in the earlier parts of this chapter (3.7) has been neglected. In fact, if we perform the same procedure in the original frame [2], the angular velocity of the dust observed by the locally nonrotating observer is

$$\omega_0' = \frac{d\varphi_0}{dt} + \frac{k}{l} \tag{3.44}$$

where $\frac{d\varphi_0}{dt}$ refers to the coordinate velocity of dust in the original frame, which is Ω. But one reminds himself that,

$$K = k + \Omega l \qquad L = l \tag{3.45}$$

which gives,

$$\omega_0' = \Omega + \frac{K - \Omega L}{L} = \frac{K}{L} = \omega \tag{3.46}$$

the same as in the comoving frame. In fact, there are infinitely many frames that differ by a coordinate transformation of the form (Eq. 3.7), and they may suggest a corresponding coordinate velocity Ω of the dust particles. But all these are just coordinate transformations; the angular velocity that the dust particles appear to be moving relative to a locally nonrotating observer, is a combination of the coordinate velocity and that of the rotation of space-time, and is always $\frac{K}{L}$. So Ω is rather a manifestation of the choice of coordinates while $\frac{K}{L}$ is intrinsic to the system. Newtonian methods would not capture this angular velocity due to space-time dragging.

For the solutions we are going to study, the expression for angular velocity of the dust is chosen to be $\frac{K}{L}$. This expression is used in Cooperstock & Tieu's solution as well in Balasin & Grumiller's. It gives sufficient insight into the rotation of the dust.

For Van Stockum's solution, specifically, the angular velocity

$$\omega = \frac{K}{L} = \frac{\alpha}{1 - \alpha^2 r^2} \tag{3.47}$$

indicates a surface of singularity at $r = \frac{1}{|\alpha|}$ where ω is divergent. This surface is an analogy

[2]denoted by f, k, and l

to the edge of the disk using Newtonian methods, which also has angular velocity blowing up to infinity [10].

3.3.2 On-Axis Angular Velocity

As discussed, there are infinitely many frames that differ from one another only through a transformation of the form $\varphi \to \Omega + \varphi t$. Each of these frames subscribes a coordinate velocity for the dust particles and this coordinate velocity is in general different. We might want to pick a particular frame among all these physically equivalent ones such that it is consistent with our observation on earth. Suppose we obtain a metric in an arbitrary frame,

$$ds^2 = f dt^2 - 2k d\varphi dt - l d\varphi^2 - e^{\mu}(dr^2 + dz^2) \tag{3.48}$$

then $\omega = \frac{k}{l}$, consistent with Eq. 3.41, will be the space-time dragging component in the angular velocity of the dust relative to a locally nonrotating observer. If this observer is at infinity (on earth), the metric should reduce to Minkowski, and consequently, this GR effect component would be 0.

However, we do not have the luxury in most cases to set $\frac{k}{l} = 0$ at infinity; instead, we impose the boundary condition of a **locally nonrotating axis** whereby the space-time dragging effect vanishes on the axis. This is how Vishveshwara & Winicour fixed one of the parameters in their solution ([17], section 2) and can be generalized to fix parameters or reduce degrees of freedom in other solutions. In the course of this project, we also encountered a similar formulation of this boundary condition (i.e., GR effects vanish on the axis) by Islam ([9], section 4.4) through a locally norotating observer on axis. Due to the fact that Vishveshwara & Winicour's formulation is more generic and simpler, we will only discuss Vishveshwara's formulation here, and leave Islam's in the appendix (A.3).

This subsection will take Van Stockum's solution as an illustration. One will find, after reading A.3 and comparing that to the formulation below, these two are equivalent in Van Stockum's solution.

Vishveshwara & Winicour demand that the space-time dragging stops on the axis,

$$\left.\frac{k}{l}\right|_{r=0} = 0 \tag{3.49}$$

which is equivalent to the requirement that what is observed by a locally nonrotating observer on the axis is equal to the coordinate velocity of the dust. This gives

$$\left.\left(\frac{\alpha}{1-\alpha^2 r^2} - \Omega\right)\right|_{r=0} = 0 \quad \Rightarrow \quad \alpha = \Omega \tag{3.50}$$

We could say that we have identified a particular frame under our condition of a locally nonrotating axis, if we know the metric components. Alternatively, we have identified one of the metric component parameters, if we know the angular velocity at which dust particles rotate with respect to the locally nonrotating observer on the axis.

Notice that in this frame the dust particles follow a four-velocity as defined in Eq. 2.10 and the metric is simply,

$$f = 1 + \Omega^2 r^2 + \Omega^4 r^4 \qquad k = \Omega^3 r^4 \qquad l = r^2(1 - \Omega^2 r^2) \tag{3.51}$$

in this way we fix α in Van Stockum's solution.

3.4 Bonnor's Solution

Bonnor started with a harmonic function v of the form

$$v = \frac{2\alpha}{\sqrt{z^2 + r^2}} \tag{3.52}$$

and derived the following solution using Eq.3.13:

$$K = -2\alpha \frac{r^2}{(z^2 + r^2)^{\frac{3}{2}}} \tag{3.53}$$

The rest of the quantities follow,

$$L = r^2 - K^2 \tag{3.54}$$

$$\mu = \frac{1}{2}\alpha^2 r^2 \frac{(r^2 - 8z^2)}{(z^2 + r^2)^4} \tag{3.55}$$

$$mn = \frac{1}{2\pi} e^{-\mu} \alpha^2 \frac{4z^2 + r^2}{(z^2 + r^2)^4} \tag{3.56}$$

This solution has a few properties more physical than Van Stockum's. First, with a dependence on z, this solution is not cylindrical anymore. It does not extend to both infinities in the axis direction. Second, in terms of its asymptotic behavior, when $r \to \infty$ or $z \to \infty$,

$$K \to 0 \quad L \to r^2 \quad \mu \to 0 \quad mn \to 0 \tag{3.57}$$

so the metric becomes Minkowski. However, as r and z both goes to zero, we realize that many of the components of the metric, namely, K, L and μ, become divergent. This suggests a singularity point at the center.

The more serious problem of this singularity is not its existence, but rather its property, if we look at the Newtonian limit of this solution. Transforming into the locally nonrotating frame,

$$ds^2 = \left(1 + \frac{K^2}{L}\right) dt^2 - L d\varphi^2 + e^{\mu}(dr^2 + dz^2) \tag{3.58}$$

where we can expand g_{00} as

$$1 + \frac{K^2}{L} = 1 + \frac{4\alpha^2 r^2}{R^6 - 4\alpha^2 r^2} = 1 + \frac{4\alpha^2 R^2 \cos^2\theta}{R^6 - 4\alpha^2 R^2 \cos^2\theta} = 1 + O\left(\frac{1}{R^4}\right) \tag{3.59}$$

We can confirm by this equation that it does not contain any $O(\frac{1}{R})$ term.

The asymptotic flatness suggests that the gravitation becomes a weak field at a far distance, i.e. at R very large, one can study the Newtonian limit of this solution by looking at its linearised metric as an approximation. Referring to the expression of the mass M of a relativistic body [15], we have

$$g_{00} = 1 + 2\Phi = 1 - \frac{2M}{R} + O\left(\frac{1}{R^2}\right) \tag{3.60}$$

where Φ is the Newtonian gravitational field. Clearly, reading from the expansion of Bonnor's solution (Eq. 3.59), one realizes that there is no mass observed at infinity (R very large). This is at contradiction with the mass density calculated in Eq. 3.56 as it is well defined and positive everywhere outside of the origin. Such a contradiction points to a negatively divergent mass at the center where the singularity occur. It is again not a physical solution.

3.5 Cooperstock & Tieu's Model

Cooperstock & Tieu claim that their solution models rotation curves of galaxies aptly. Using this model, the tangential velocity of the dust can be fit with the observational data in good agreement (Fig. 1.2).

In this section, we will discuss the basic structure of the solution, some approximations involved and the implications.

The solution adopts a method that is equivalent to Eq. 3.13 to construct K

$$K = rv_r(r, z) \tag{3.61}$$

Further, separation of variables and linear combination of modes is used for the harmonic function $v(r, z)$.

$$v(r, z) = R(r)Z(z) \tag{3.62}$$

$$R_{rr}Z + RZ_{zz} + \frac{R_r}{r}Z = 0 \tag{3.63}$$

which is compactly written as,

$$\frac{Z_{zz}}{Z} = -\frac{\frac{R_r}{r} + R_{rr}}{R} = k \tag{3.64}$$

for some constant k. Referring to the properties of exponential functions and Bessel functions (A.4), Eq. 3.64 can be solved by

$$Z = e^{\pm\sqrt{k}z} \tag{3.65}$$

$$R = J_0(\sqrt{k}r) \tag{3.66}$$

up to some integration constants. $J_0(\sqrt{k}r)$ is the Bessel function of the first kind with an order of 0. One further demands that Z is an even function because the galaxy should be symmetric about the galactic plane, thus

$$Z = e^{-\sqrt{k}z} + e^{\sqrt{k}z} \tag{3.67}$$

This choice of Z, however, has the first term diverging for large $z < 0$ and the second diverging for large $z > 0$. One way to circumvent this is to use a piecewise function

$$Z = e^{-\sqrt{k}|z|} \tag{3.68}$$

and arriving at the expression for a single mode n of $v(r, z)$,

$$v_n(r, z) = C_n e^{-\sqrt{k_n}|z|} J_0(\sqrt{k_n}r) \tag{3.69}$$

A linear combination gives

$$v(r, z) = \sum_n C_n e^{-\sqrt{k_n}|z|} J_0(\sqrt{k_n}r) \tag{3.70}$$

Based on the recurrence relation of Bessel functions (Appendix A.4),

$$K(r, z) = \sum_n \sqrt{k_n} C_n e^{-\sqrt{k_n}|z|} J_1(\sqrt{k_n}r) \tag{3.71}$$

has many degrees of freedom for the fitting. It is also asymptotically flat.

Cooperstock & Tieu noted in their paper [8] that the first 10 integer modes of n, starting from $n = 1$, suffice to obtain a satisfactory fitting. While it is not clear if this many degrees of freedom is an over-fitting, and it remains a question on how to test the fitting sufficiently against such a small data set; a more physical concern is with the discontinuity at the galactic plane. A revisit to the expression of mass density in Eq. 3.29 gives,

$$\frac{\partial mn}{\partial z} \propto \sum_n \frac{\partial}{\partial z} \frac{(\frac{\partial}{\partial r} K_n)^2 + (\frac{\partial}{\partial z} K_n)^2}{r^2 e^\mu} \tag{3.72}$$

and K does not have a continuous first derivative along z at any point r_0,

$$\left(\frac{\partial}{\partial z} K_n\right)\bigg|_{z\to 0^+} = -\sqrt{k_n} R(r_0) \qquad \left(\frac{\partial}{\partial z} K_n\right)\bigg|_{z\to 0^-} = \sqrt{k_n} R(r_0) \tag{3.73}$$

indicating that mn has a jump at $z = 0$ as well. As dust particles are not supposed to interact with each other, this discontinuity violates the dust assumption.

On the other hand, one should examine the expressions used in the plotting, which was taken to be

$$V(r) = -\sum_n k_n C_n e^{-\sqrt{k_n}|z|} J_1(\sqrt{k_n} r) \tag{3.74}$$

An important assumption here is that r is large compared to K, such that

$$\omega(r) r = \frac{K}{L} r = \frac{K}{r^2 - K^2} r \simeq \frac{K}{r^2} r = \frac{K}{r} \tag{3.75}$$

This assumption should not hold for situations close to the axis.

Cooperstock & Tieu did not include an analytical analysis of the density profile, here a simple sketch of the proof that the density converges is given.

We consider only one term in the summation, that is, $K = rR_rZ$ with $k = k_n$. As a reminder of the previous results, the following is the formula for density derived in the first section of this chapter (Eq. 3.29),

$$mn \propto \frac{{K_r}^2 + {K_z}^2}{r^2 e^\mu} \tag{3.76}$$

and

$$\mu_r = \frac{{K_z}^2 - {K_r}^2}{2r} \quad \mu_r = -\frac{K_r K_z}{r} \tag{3.77}$$

where the sign of the constants is important.

One first focuses on the term $\frac{{K_r}^2+{K_z}^2}{r^2}$

$$K_r = R_r Z + r R_{rr} Z = \frac{Z}{r}(r^2 R_{rr} + r R_r) = -k_n \frac{Z}{r} r^2 R = -r k_n Z R \tag{3.78}$$

$$K_z = r R_r Z_z = -sgn(z) r \sqrt{k_n} R_r Z \tag{3.79}$$

$$\frac{{K_r}^2 + {K_z}^2}{r^2} = {k_n}^2 R^2 Z^2 + k_n R_r^2 Z^2 = k_n Z^2 (k_n R^2 + {R_r}^2) \tag{3.80}$$

We know the asymptotic behavior of Bessel functions:

$$\begin{aligned} R(r) &\propto J_0(\sqrt{k_n} r) \to \sqrt{\frac{2}{\pi \sqrt{k_n} r}} \cos\left(\sqrt{k_n} r - \frac{\pi}{4}\right) \\ R_r(r) &\propto J_0(\sqrt{k_n} r) \to \sqrt{\frac{2}{\pi \sqrt{k_n} r}} \cos\left(\sqrt{k_n} r - \frac{3\pi}{4}\right) \end{aligned} \tag{3.81}$$

which converge to 0 at infinity. Also, Z is exponentially decaying, so $\frac{{K_r}^2+{K_z}^2}{r^2}$ certainly goes to 0.

For the $\frac{1}{e^\mu}$ term, one can integrate μ explicitly, since

$$\mu_r = -\frac{r}{2} Z^2 k_n (k_n R^2 - R_r^2) \tag{3.82}$$

$$\mu_z = -sgn(z) r Z^2 {k_n}^{\frac{3}{2}} R R_r \tag{3.83}$$

The second equation can be integrated with respect to z, producing

$$\mu = \frac{k_n}{2} r R R_r Z^2 + \nu(r) \equiv \mu_0 + \nu(r) \tag{3.84}$$

We can deduce the $\nu(r)$ term through

$$\frac{\partial(\mu_0 + \nu(r))}{r} = \mu_r \tag{3.85}$$

$$\frac{k_n}{2}(R R_r + r R_r^2 + r R R_{rr}) Z^2 + \frac{d\nu(r)}{dr} = -\frac{r}{2} Z^2 k_n (k_n R^2 - R_r^2) \tag{3.86}$$

which gives,

$$\nu(r) = C_n \tag{3.87}$$

a constant, as one simplifies the left-hand side of Eq. 3.86 using Eq. 3.64. Therefore, one can write down the explicit form of μ as

$$\mu = \frac{k_n}{2} r R R_r Z^2 \propto \cos\left(\sqrt{k_n} r - \frac{\pi}{4}\right) \cos\left(\sqrt{k_n} r - \frac{3\pi}{4}\right) e^{-2\sqrt{k}|z|} + C_n \tag{3.88}$$

This term is bounded above and below. Since we have shown that $\frac{K_r^2 + K_z^2}{r^2}$ goes to 0 at infinity, this completes the argument for the claim that the density profile goes to zero at infinity.

3.6 "Rod" Solution

During the course of this project, we also tried a solution inspired by that of Balasin & Grumiller (section 3.7). It takes a simpler form than theirs (Eq. 3.106) but captures the essence of the solution rather well.

We start from constructing K using the second method (Eq. 3.14), and with a u of the form

$$u = a \log \frac{-m + z + \sqrt{(-m+z)^2 + r^2}}{m + z + \sqrt{(m+z)^2 + r^2}} \tag{3.89}$$

which is the gravitational field generated by a homogeneous and infinitely thin rod of density a on the z axis, spanning from $-m$ to m. The ∂_r and ∂_z derivatives of u give the correct magnitude for forces in such a gravitational field,

$$\frac{\partial u}{\partial r} = \left. \frac{ar}{\sqrt{(z_0 - z)^2 + r^2}(z - z_0 + \sqrt{(z - z_0)^2 + r^2})} \right|_{z=-m}^{z=m} \tag{3.90}$$

$$= \int_{-m}^{m} \frac{dz_0 ar}{((z - z_0)^2 + r^2)^{\frac{3}{2}}} \tag{3.91}$$

$$\frac{\partial u}{\partial z} = \left. \frac{a}{\sqrt{((z_0 - z)^2 + r^2)}} \right|_{z_0=-m}^{z_0=m} \tag{3.92}$$

$$= \int_{-m}^{m} \frac{dz_0 a(z - z_0)}{((z - z_0)^2 + r^2)^{\frac{3}{2}}} \tag{3.93}$$

Now, to simplify the expression, one could use the following symbols

$$R_+ = \sqrt{(m + z)^2 + r^2} \tag{3.94}$$

$$R_- = \sqrt{(-m + z)^2 + r^2} \tag{3.95}$$

and it follows that,

$$K = a(R_- - R_+) \tag{3.96}$$

$$L = r^2 - a^2(R_- - R_+)^2 \tag{3.97}$$

$$\mu = -a^2 \log R_- R_+ - a^2 \log(\sqrt{(R_+^2 - r^2}\sqrt{R_-^2 - r^2} + R_+ R_-) \tag{3.98}$$

The density function has a very complicated form, therefore we will only discuss its values on the equatorial plane and on the axis,

$$mn(r, 0) \propto \frac{(m^2 + r^2)^{a^2 - 1}}{r^{2a^2 + 2}} \tag{3.99}$$

which is asymptotically flat, but divergent on the axis.

The most salient problem of this solution, is that it has the wrong symmetry. To see this, one can transform the result to be in spherical coordinates via

$$r \to \sqrt{\rho^2 - 2m\rho} \sin\theta \tag{3.100}$$

$$z \to (\rho - m)\cos\theta \tag{3.101}$$

that is asymptotically spherical, and obtain

$$K = a(\sqrt{(m - \rho + m\cos\theta)^2} - \sqrt{(\rho - m + m\cos\theta)^2}) \tag{3.102}$$

To simplify, we observe that when $\rho \geq 2m$,

$$K = a(-m + \rho - m\cos\theta) - a(\rho - m + m\cos\theta) = -2am\cos\theta \tag{3.103}$$

which suggests that the solution is not symmetric about the equatorial plane outside of the surface $r = 2m$. Specifically, K at the north pole ($\theta = 0$) and the south pole ($\theta = \pi$) are negatives of each other, as if the system is twisted,

$$K|_{\theta=0} = 2am \qquad K|_{\theta=\pi} = -2am \tag{3.104}$$

This certainly should not be the case, given that the sign of K implies the direction of rotation, and it is strange to have a galaxy that rotates at different directions on the opposite poles. In short, it is obvious that on the equatorial plane ω is zero, as indicated by Eq. 3.102; on the axis, ω is undefined when z goes to $\pm\infty$, back to cylindrical coordinates,

$$\omega|_{z=0} = 0 \qquad \omega|_{r=0} = \frac{1}{a(\sqrt{(m+z)^2} - \sqrt{(m-z)^2})} \tag{3.105}$$

This solution, though not physical, is going to serve as a precursor to what we are going to observe in Balasin & Grumiller's case, which does not have irregularities at the first sight.

3.7 Balasin & Grumiller's Model

Superposition of two rods, one from R to r_0 and another from $-r_0$ to $-R$ (without the loss of generality, one can assume that $R > r_0 > 0$), gives rise to the following K,

$$K(r, z) = V_0(R - r_0) + \frac{V_0}{2}\sum_{\pm}\left(\sqrt{(z \pm r_0)^2 + r^2} - \sqrt{(z \pm R)^2 + r^2}\right) \tag{3.106}$$

Notice that the orientation of the two rods is opposite to each other, and that both are shifted away from the origin. This configuration saves the solution from being twisted. As an illustration, we plot K against r with some arbitrary constants (Fig. 3.1).

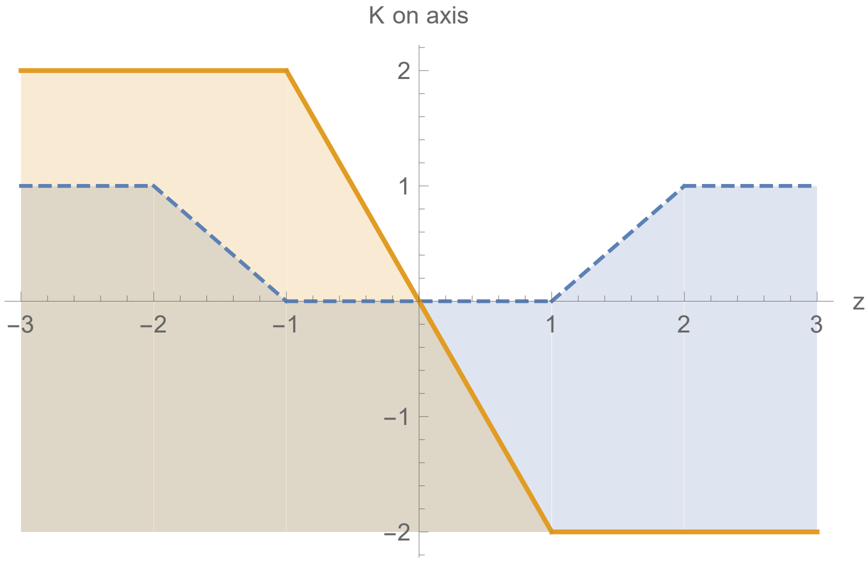

Figure 3.1: Thick curve: K in single rod solution on the axis (Eq. 3.96 at $r = 0$, with constants: $a = 1$, $m = 1$). Dashed curve: K in Balasin & Grumiller's superpositioned rods solution on the axis (Eq. 3.106 at $r = 0$, with constants $V_0 = 1$, $r_0 = 1$, $R = 2$).

Like Cooperstock & Tieu, Balasin & Grumiller claimed that this solution gives a satisfactory curve at $z = 0$ as well (Fig: 3.2).

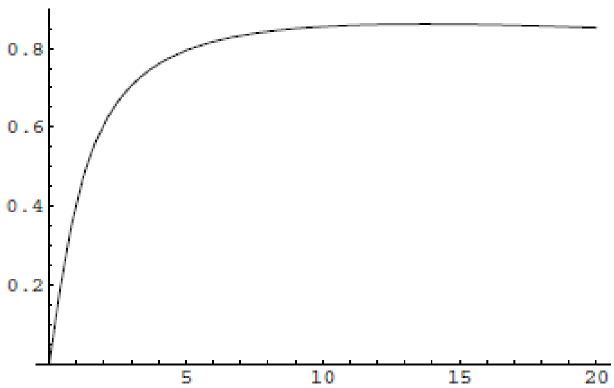

Figure 3.2: Plot of $\frac{V}{V_0}|_{z=0}$ from Balasin & Grumiller's paper, with constants $V_0 = 1$, $r_0 = 1$, $R = 100$, as required for the fitting [1].

Clearly, the tangential velocity has a nice plateau that we are trying to find. However, similar to what we discussed in section 3.5, the approximation that

$$\omega = \frac{K}{L} = \frac{K}{r^2 - K^2} \simeq \frac{K}{r^2} \tag{3.107}$$

is not valid close to the axis. In fact, the difference between the two expressions is not negligible (Fig. 3.3).

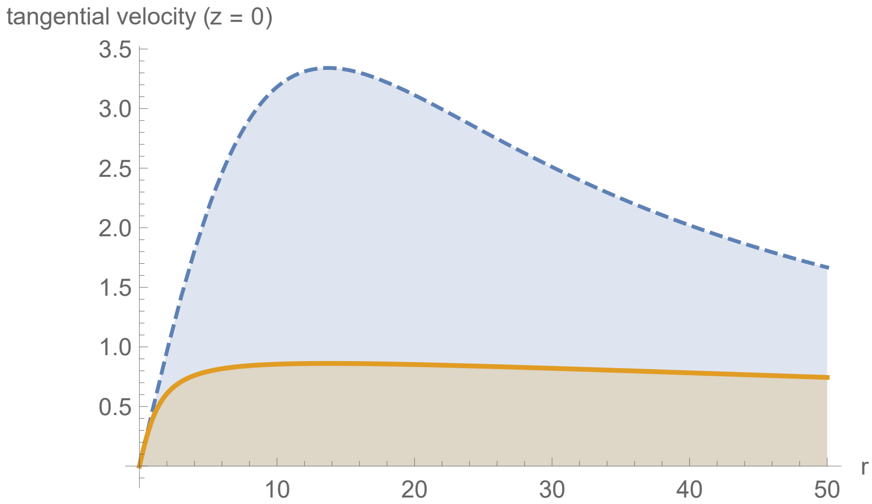

Figure 3.3: Difference between the exact expression $\frac{K}{L}$ (dashed curve) and the approximated $\frac{K}{r^2}$ (thick curve). The plot is obtained for the equatorial plane where $z = 0$. Constants: $V_0 = 1$, $r_0 = 1$, $R = 100$, as required for the fitting in Fig. 3.2.

On the other hand, in the expression for the density mn, the approximation

$$mn = \frac{K_r^2 + K_z^2}{r^2 e^\mu} \simeq \frac{K_r^2 + K_z^2}{r^2} \tag{3.108}$$

is applied. Analytically, it is difficult to write down the expression for μ, but we have the simplified single rod solution as an analogy,

$$\mu|_{z=0} = a^2 \log \frac{2r^2}{m^2 + r^2} \tag{3.109}$$

When r is small, the argument in the log function is close to 0, and μ goes to negative infinity. $e^{-\mu}$ will blow up. This clearly indicates that the approximation in Eq. 3.108 also breaks down close to the rotational axis.

On another note, if one possibly assumes that the approximations are valid, this solution is still **interior**, as pointed out by Balasin & Grumiller themselves. Two concerns exist. Firstly, there are two singularity curves along which the tangential velocity diverges. Secondly, density is also divergent along the axis (Fig. 3.4),

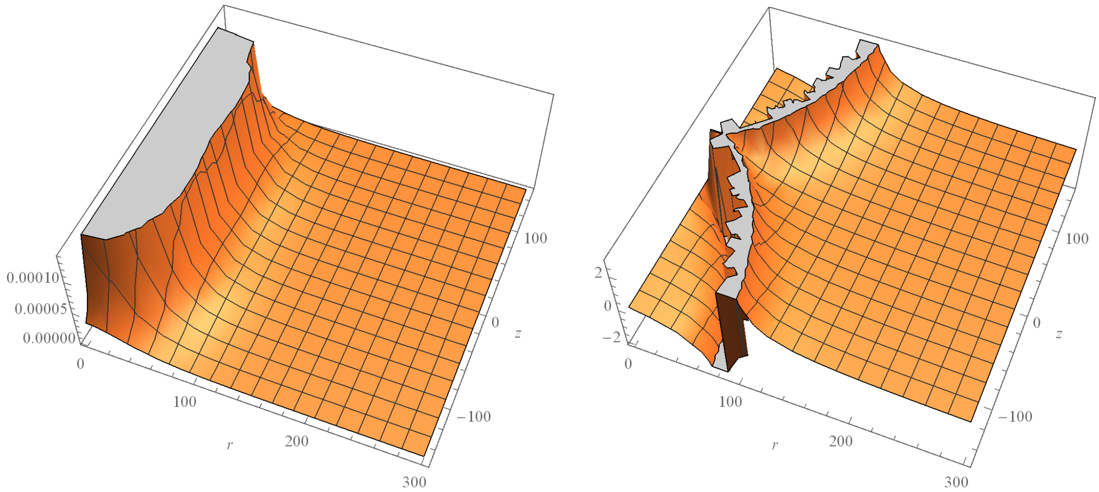

Figure 3.4: Density (left) and tangential velocity (right) in Balasin & Grummiler's soluion, with constants $V_0 = 1$, $r_0 = 1$, $R = 100$, as required for the fitting in [1].

To these two concerns, Balasin & Grumiller attribute the divergence of density on the axis to jets in the galactic center, and restricted the applicability of their solution to only the interior region of the galaxy, i.e., in between the singularity curves.

If one ignores the region near the axis, then Balasin & Grumiller's solution is a good interior solution. Yet once we enter this region, the two assumptions will break down and the nice plateau is gone. There might exist some singularity surface, but an analytical expression for that is complicated.

Chapter 4

General Relativistic Dust Models: Differential Rotation

The differentially rotating dust model is more complicated and possesses one more degree of freedom compared to that of the rigidly rotating dust model (Recall, that Eq. 3.12 is the only independent equation to solve, the rest of the metric components follow). Now that Ω is no longer a constant, Eq. 3.5 will not imply that F is a constant. Therefore, the approach should differ thereafter. Again, we follow Islam's deriviation, and interested readers can refer to Winicour [20] for the original derivation.

4.1 General Approach

Writing Eq. 3.5 without the assumption that Ω is a constant, one arrives at

$$F_r + 2\Omega_r K = 0 \tag{4.1}$$

$$F_z + 2\Omega_z K = 0 \tag{4.2}$$

Multiplying Eq. 4.1 by Ω_z and Eq. 4.2 by Ω_r, one obtains,

$$\Omega_z F_r - \Omega_r F_z = 0 \tag{4.3}$$

One then deduces that

$$F = F(\Omega) \tag{4.4}$$

is a solution. To see this, we assume that F is of the form,

$$F = F(r, \Omega) \tag{4.5}$$

and Ω itself a function of r and z of course. It then follows that,

$$0 = \Omega_z F_r - \Omega_r F_z = \Omega_z \left(\frac{\partial F}{\partial r}\right)\bigg|_{\Omega} + \Omega_z\Omega_r \left(\frac{\partial F}{\partial \Omega}\right)\bigg|_{r} - \Omega_z\Omega_r \left(\frac{\partial F}{\partial \Omega}\right)\bigg|_{r} \tag{4.6}$$

$$= \Omega_z \left(\frac{\partial F}{\partial r}\right)\bigg|_{\Omega} \tag{4.7}$$

and since Ω_z should not be generally 0, $\left(\frac{\partial F}{\partial r}\right)|_{\Omega}$ is. This is equivalently saying that F does not depend explicitly on r. This conclusion can be derived in a similar fashion if one assumes that $F = F(z, \Omega)$.

Now with Eq. 4.1 and 4.2,

$$\frac{dF}{d\Omega} = -2K \equiv F' \tag{4.8}$$

and take note that F' does not refer to the locally nonrotating frame in the previous chapter, but the derivative of F with respect to Ω. Since Eq. 3.11 is still valid, one can derive that,

$$\begin{aligned} 0 &= (k + \Omega l)R_{00} + (f + \Omega^2 l)R_{03} + \Omega(f - \Omega k)R_{33} \\ &= K\Delta F - F\Delta K + 4K(K_r\Omega_r + K_z\Omega_z) \\ &\quad - 2KL(\Omega_r^2 + \Omega_z^2) + (2K^2 + LF)\Delta\Omega + 2F(L_r\Omega_r + L_z\Omega_z) \end{aligned} \tag{4.9}$$

To simplify this, we might look at some identities in the new formulation first. Notice that since $FL + K^2 = r^2$ still holds in the rotating frame [1] so that, by taking derivatives

[1] $FL + K^2 = (f - 2\Omega k - \Omega^2 l)l + (k + \Omega l)^2 = fl - 2\Omega kl - \Omega^2 l^2 + k^2 + 2\Omega kl + \Omega^2 l^2 = fl + k^2$

of r and z on both sides of this equation,

$$F'\Omega_r L + FL_r + 2KK'\Omega_r = 2r \tag{4.10}$$

$$\Rightarrow FL_r = 2r - F'L\Omega_r - 2KK'\Omega_r \tag{4.11}$$

$$F'\Omega_z L + FL_z + 2KK'\Omega_z = 0 \tag{4.12}$$

$$\Rightarrow FL_z = -F'L\Omega_z - 2KK'\Omega_z \tag{4.13}$$

again K' denotes the derivative of K with respect to Ω. Another two identities important to simplify Eq. 4.9 are

$$\Delta F = F'\left(\Omega_{rr} + \Omega_{zz} - \frac{1}{r}\Omega_r\right) + F''(\Omega_r^2 + \Omega_z^2) = F'\Delta\Omega + F''(\Omega_r^2 + \Omega_z^2) \tag{4.14}$$

$$\Delta K = K'\left(\Omega_{rr} + \Omega_{zz} - \frac{1}{r}\Omega_r\right) + K''(\Omega_r^2 + \Omega_z^2) = K'\Delta\Omega + K''(\Omega_r^2 + \Omega_z^2) \tag{4.15}$$

Therefore, Eq. 4.9 is simplified to be

$$\Delta\Omega(KF' - FK' + r^2 + K^2) + (\Omega_r^2 + \Omega_z^2)(KF'' - FK'' - \frac{F'}{F}(r^2 - K^2)) + 4r\Omega_r = 0 \tag{4.16}$$

Looking at Eq. 4.14 and 4.15, where Δ of a function of Ω is written in terms of $\Delta\Omega$ and $\Omega_r^2 + \Omega_z^2$, we then try to write eq 4.16 also as a Δ of some function of Ω, for example, ζ. Notice that there is an extra Ω_r term, which in ΔF was absorbed by the derivative of F' with respect to r and made into a ${\Omega_r}^2$ term. If this function ζ is explicitly also dependent of r, then the term will not be absorbed. In short, we have a function $\zeta(r,\Omega)$ that satisfies $\Delta\zeta = 0$, and it should give us an expansion in the form of Eq. 4.16.

We first get the notations correct.

$$\dot{\zeta} = \left(\frac{\partial\zeta}{\partial r}\right)\Bigg|_{\Omega} \qquad \zeta' = \left(\frac{\partial\zeta}{\partial\Omega}\right)\Bigg|_{r} \tag{4.17}$$

and expand $\Delta\zeta$ in the form of Eq. 4.16,

$$\Delta\zeta = \zeta'\Delta\Omega + \zeta''(\Omega_r^2 + \Omega_z^2) + 2\dot{\zeta}'\Omega_r + \left(\frac{\dot{\zeta}}{r} + \ddot{\zeta}\right) \tag{4.18}$$

We must multiply the expression in Eq. 4.16 by $\frac{1}{F}$ to ensure that the term in front of $\Delta\Omega$, which will be our ζ' gives the term in front of $({\Omega_r}^2 + {\Omega_z}^2)$ when taking the derivative of Ω while keeping r constant. Meanwhile, it satisfies the matching of $\dot{\zeta}'$ with the term in front of Ω_r and that $\frac{\zeta'}{r} + \ddot{\zeta} = 0$. Luckily, all can be simultaneously achieved if

$$\zeta = \int \frac{1}{F}(KF' - FK' + r^2 + K^2)d\Omega \tag{4.19}$$

which is equivalently

$$\zeta' = \frac{1}{F}(KF' - FK' + r^2 + K^2) \tag{4.20}$$

$$\zeta'' = \frac{1}{F}\left(KF'' - FK'' - \frac{F'}{F}\left(r^2 - K^2\right)\right) \tag{4.21}$$

$$\dot{\zeta}' = \frac{2r}{F} \tag{4.22}$$

$$\ddot{\zeta} - \frac{1}{r}\dot{\zeta} = 0 \tag{4.23}$$

It is readily verified that Eq. 4.20 and 4.21 are consistent, since

$$\begin{aligned}\left.\left(\frac{\partial\zeta'}{\partial\Omega}\right)\right|_r &= \frac{1}{F}\left(K'F' + KF'' - F'K' - FK'' + 2KK'\right) - \frac{F'}{F^2}\left(KF' - FK' + r^2 + K^2\right) \\ &= \frac{1}{F}\left[kF'' - FK'' + 2KK' - \frac{F'}{F}(KF' + r^2 + K^2) + F'K'\right] \\ &= \frac{1}{F}\left(KF'' - FK'' - \frac{F'}{F}(r^2 - K^2)\right) = \zeta''\end{aligned} \tag{4.24}$$

where $-2K = F'$ was used in the second line to obtain the third.

For the consistency between Eq. 4.20 and 4.22, it is simple as F and K are single variable functions of Ω and one quickly sees that the derivative of ζ' with respect to r while keeping Ω constant would only be differentiating the $\frac{r^2}{F}$ term. For Eq. 4.23, we can assume that ζ is of the form

$$\zeta \equiv G(\Omega) + r^2 H(\Omega) \tag{4.25}$$

from Eq. 4.19. Consequently,

$$\ddot{\zeta} - \frac{1}{r}\dot{\zeta} = H(\Omega)\left(2 - \frac{1}{r}2r\right) = 0 \tag{4.26}$$

is consistent with Eq. 4.23.

We have now specified the two degrees of freedom, one is the choice of $F(\Omega)$ and the other $\zeta(r, \Omega)$. One can first choose $F(\Omega)$ and substitude that into Eq. 4.19, arriving at an expression of ζ in terms of r and Ω. One then chooses the form of ζ that satisfies $\Delta\Omega = 0$, and will have an equation that involves only Ω, r, and z. Ω can then be determined implicitly.

Finally, we have the three equations that determine the coefficient μ and the mass density term $8\pi mn$, dervied in the same vein as in Eq. 3.28 and 3.29

$$\begin{aligned}\mu_r = -\frac{1}{2r}(F_r L_r + {K_r}^2 - F_z L_z - K_z^2 \\ + 2(KL_r - LK_r)\Omega_r - 2(KL_z - LK_z)\Omega_z + L^2(\Omega_r^2 - \Omega_z^2))\end{aligned} \tag{4.27}$$

$$\begin{aligned}\mu_z = -\frac{1}{2r}(F_r L_z + F_z L_r + 2K_r K_z + \\ 2(KL_z - LK_z)\Omega_r + 2(KL_r - LK_r)\Omega_z + 2L^2\Omega_r\Omega_z)\end{aligned} \tag{4.28}$$

$$8\pi mne^{\mu} = \frac{1}{F}(\Delta F + 2(2K' - L)(\Omega_r^2 + \Omega_z^2) + 2K\Delta\Omega) + \frac{\Sigma}{r} \tag{4.29}$$

where

$$\Sigma = F_r L_r + F_z L_z + K_r^2 + K_z^2 + 2\Omega_r(KL_r - LK_r) + 2\Omega_z(KL_z - LK_z) + L^2(\Omega_r^2 + \Omega_z^2) \tag{4.30}$$

4.2 Vishveshwara & Winicour's solution

Vishveshwara & Winicour [17] took a different route to solve for differentially rotating dust, but their solution can be identified as follows,

$$F(\Omega) = -\left(1 + \frac{(\Omega - \Omega_0)^2}{p^2}\right) \qquad \zeta = \Omega_0 r^2 \tag{4.31}$$

where p and Ω_0 are constants to be determined, and as promised, F is a single variable function of Ω and ζ satisfies the equation $\Delta\zeta = 0$. This yields a K and Ω of the form

$$K = \frac{\Omega - \Omega_0}{p^2} = \frac{1}{p}\tan\frac{\Omega_0 p r^2}{1+p^2r^2} \tag{4.32}$$

$$\Omega = \Omega_0 - p\tan\left(\frac{\Omega_0 p r^2}{1+p^2r^2}\right) \tag{4.33}$$

and the density is

$$8\pi mne^{\mu} = 4\frac{{\Omega_0}^2(1-p^2r^2)}{(1+p^2r^2)} \tag{4.34}$$

Notice that this is very similar to the solution of Van Stockum's (section 3.2). First of all, it is independent of z, hence a cylinder extending in both directions to infinity; secondly, when p is small compare to 1, Ω and K are approximated to be,

$$\Omega = \Omega_0 + p\tan\frac{\Omega_0 p r^2}{1+p^2r^2} \simeq \Omega_0 + \Omega_0\frac{p^2r^2}{1+p^2r^2} \simeq \Omega_0 \tag{4.35}$$

$$K = \frac{\Omega - \Omega_0}{p^2} = \frac{1}{p}\tan\frac{\Omega_0 p r^2}{1+p^2r^2} \simeq \frac{\Omega_0 r^2}{1+p^2r^2} \simeq \Omega_0 r^2 \tag{4.36}$$

in cylindrical coordinates. An important point in obtaining this form of solution is by demanding the space-time dragging component $\frac{k}{l} = 0$ to vanish on the axis, as the boundary condition mentioned in section 3.3.2.

It is clear that this solution cannot be a physical solution without a z dependence. Still, the tangential velocity profile does possess some desired properties. Writing

$$\frac{\Omega_0 p r^2}{1+p^2r^2} \equiv \theta \tag{4.37}$$

as a simplification, then the tangential velocity can be expressed as,

$$v = r\frac{K}{L} = \frac{\tan\theta}{\cos^2\theta}\frac{pr}{p^2r^2 - \tan^2\theta} \tag{4.38}$$

At the two extremes of r,

$$r \ll 1 \qquad \theta \simeq \Omega_0 p r^2 \ll 1 \quad \Rightarrow \quad v \simeq \frac{\Omega_0 r}{1 - {\Omega_0}^2} \tag{4.39}$$

$$r \gg 1 \qquad \theta \simeq \frac{\Omega_0}{p} \quad \Rightarrow \quad v \propto \frac{1}{pr} \tag{4.40}$$

which means that velocity grows linearly near the center, and dies off at infinity. The profile could permit a narrow plateau, as can be seen in the plot below (Fig. 4.1). Notice that since none of the metric terms are z dependent, the following graphs would be of single variable functions.

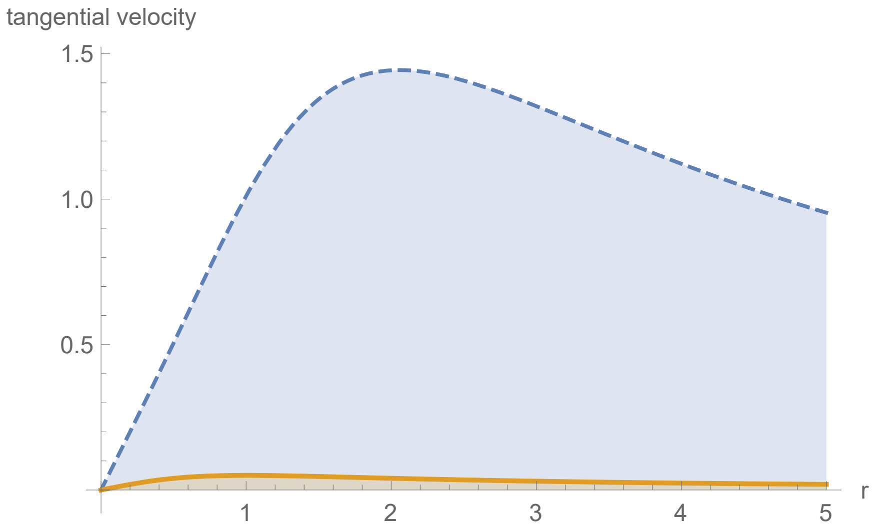

Figure 4.1: Velocity profile of Vishveshwara & Winicour's solution, with constants $\Omega_0 = 1$, $p = 1$ (dashed curve) and $\Omega_0 = 0.1$, $p = 1$ (thick curve).

Unlike Van Stockum's solution in which the tangential velocity goes proportionally to r^2, the tangential velocity in Vishveshwara & Winicour's solution is more well-behaved, if suitable parameters are chosen. However, the mass density is not as well-behaved as expected. Refer to Eq. 4.34,

$$r > \frac{1}{p} \qquad \Rightarrow \qquad 8\pi m n e^{\mu} < 0 \tag{4.41}$$

and one could visualize this by looking at the following plot of the density of this solution,

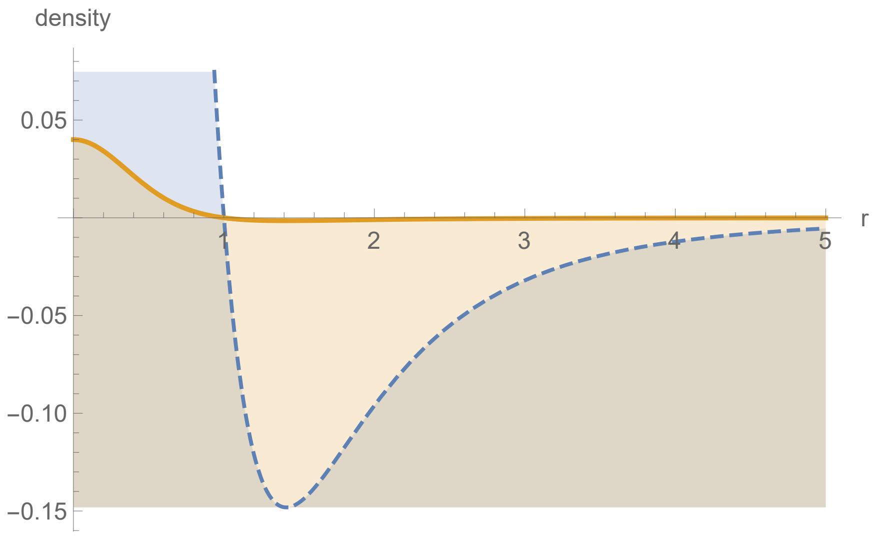

Figure 4.2: Density profile of Vishveshwara & Winicour's solution with constants $\Omega_0 = 1$, $p = 1$ (dashed curve) and $\Omega_0 = 0.1$, $p = 1$ (thick curve). Note that both densities are finite.

The negative mass density region disqualifies the solution even without the symmetry arguments.

Chapter 5

Summary

Results In this project, we have followed the general method proposed by Islam [9] in solving Einstein's equation for an axially-symmetric rotating metric with dust approximation. Several proposed solutions, from authors including Van Stockum (3.2), Bonnor(3.4), Cooperstock & Tieu (3.5), Balasin & Grumiller (3.7) and finally Vishveshara & Winicour (4.2), have been analysed in terms of their physical properties and implications. We have also constructed a solution (3.6) inspired by Balasin & Grumiller.

All solutions we have discussed so far are not perfect. They come with various characteristics at odds with what we hoped for, namely,

- Cylindrical symmetry (Van Stockum, Vishveshwara & Winicour)
- Unbounded density at infinity (Van Stockum)
- Negative mass density (Bonnor, Vishveshwara & Winicour)
- Twist ("Rod")
- Not smooth at the galactic plane (Cooperstock & Tieu) [1]
- Singularity other than on the axis (Van Stockum, Balasin & Grumiller[2])
- Narrow plateau (Balasin & Gruimiller, Vishveshwara & Winicour)

We have reproduced all these solutions from certain choices of harmonic functions, and

[1] Cooperstock & Tieu have responded to this point in section 4 of their paper [8] and this matter is still in discussion.

[2] Balasin & Grumiller have explained in their paper [1] that the solution is only interior, hence such properties are expected.

it might be possible that there is simply no such harmonic functions that could provide a perfect solution. In fact, since we had only discussed one solution in the differentially rotating case, there is still hope that the differentially rotating case could permit a satisfactory answer. This could be done in future work.

We have also seen that we can identify the angular velocity of the dust relative to a locally nonrotating observer by Eq. 3.41, which is invariant under coordinate transformation of the form Eq. 3.7. This would not be captured by the Newtonian picture.

Assumptions Revisited Looking back at the assumptions we made in Chapter 2, we might want to reconsider one of the assumptions, dust approximation, might break down near the axis. As most of the matter is concentrated at the center of the galaxy, the mass density might exceed the limit where dust approximations are valid. For example, collision and scattering will probably happen among the stars and interstellar medium when they are close to the galaxy nucleus. Even so, ignoring the singularities at the center, the solutions studied in this project have other problems, leading us to suspect that patching might be necessary; a global solution to rotating galaxies is not mathematically feasible.

Future Work There is still a lot to do on modelling galaxy rotation curves using GR. By looking at different harmonic functions, one can always arrive at a new solution of either rigid rotation (through Eq. 3.13 or 3.14) or differential rotation (through Eq. 4.18). The additional degree of freedom in the differential rotation case (Eq.4.4) even implies more possibilities.

Nevertheless, due to the complexity of Einstein's equation, trial and error is not be the best method. Some analytical conditions on the nature of the harmonic function or $F(\Omega)$ would help narrowing down the scope in which solutions are searched for.

Appendix A

Appendix

A.1 Curvature

For the metric of interest to this thesis, Eq. (2.7), quantities relating to its curvature are listed below [9].

Denotes $D^2 \equiv fl + k^2$,

$$
\begin{aligned}
R_{00} &= \frac{D}{2e^{\mu}}\left[\left(\frac{f_r}{D}\right)_r + \left(\frac{f_z}{D}\right)_z + \frac{f}{D^3}\left(f_r l_r + f_z l_z + {k_r}^2 + {k_z}^2\right)\right] \\
R_{03} &= -\frac{D}{2e^{\mu}}\left[\left(\frac{k_r}{D}\right)_r + \left(\frac{k_z}{D}\right)_z + \frac{k}{D^3}\left(f_r l_r + f_z l_z + {k_r}^2 + {k_z}^2\right)\right] \\
R_{33} &= -\frac{D}{2e^{\mu}}\left[\left(\frac{l_r}{D}\right)_r + \left(\frac{l_z}{D}\right)_z + \frac{k}{D^3}\left(f_r l_r + f_z l_z + {k_r}^2 + {k_z}^2\right)\right] \\
R_{11} &= \frac{1}{2}\left(-\mu_{rr} - \mu_{zz} + \frac{1}{r}\mu_r + \frac{1}{r^2}\left(f_r l_r + {k_r}^2\right)\right) \\
R_{22} &= \frac{1}{2}\left(-\mu_{rr} - \mu_{zz} + \frac{1}{r}\mu_r - \frac{1}{r^2}\left(f_z l_z + {k_z}^2\right)\right) \\
R_{12} &= \frac{1}{2}\left(\frac{1}{r}\mu_z + \frac{1}{2r^2}\left(f_r l_z + f_z l_r + 2k_r k_z\right)\right)
\end{aligned} \tag{A.1}
$$

And the relevant Christoffel symbols,

$$
\begin{aligned}
&\Gamma^0_{01} = \frac{1}{2}D^{-2}(lf_r + kk_r) \qquad \Gamma^0_{02} = \frac{1}{2}D^{-2}(lf_z + kk_z) \\
&\Gamma^0_{13} = \frac{1}{2}D^{-2}(kl_r - lk_r) \qquad \Gamma^0_{23} = \frac{1}{2}D^{-2}(kl_z - lk_z) \\
&\Gamma^1_{00} = \frac{1}{2e^{\mu}}f_r \qquad \Gamma^1_{03} = -\frac{1}{2e^{\mu}}k_r \\
&\Gamma^1_{11} = \frac{1}{2e^{\mu}}\mu_r \qquad \Gamma^1_{12} = \frac{1}{2}\mu_z \\
&\Gamma^1_{22} = -\frac{1}{2}\mu_r \qquad \Gamma^1_{33} = -\frac{1}{2}l_r \\
&\Gamma^2_{00} = \frac{1}{2e^{\mu}}f_z \qquad \Gamma^2_{03} = -\frac{1}{2e^{\mu}}k_z \\
&\Gamma^2_{11} = \frac{1}{2e^{\mu}}\mu_z \qquad \Gamma^2_{12} = \frac{1}{2}\mu_r \\
&\Gamma^2_{22} = -\frac{1}{2}\mu_z \qquad \Gamma^2_{33} = -\frac{1}{2}l_z \\
&\Gamma^3_{01} = \frac{1}{2}D^{-2}(fk_r - kf_r) \qquad \Gamma^3_{02} = \frac{1}{2}D^{-2}(fk_z - kf_z) \\
&\Gamma^3_{13} = \frac{1}{2}D^{-2}(fl_r + kk_r) \qquad \Gamma^3_{23} = \frac{1}{2}D^{-2}(fl_z + kk_z)
\end{aligned}
\tag{A.2}
$$

A.2 Geodesics

A.2.1 Geodesic in the comoving frame

We first consider the geodesic equation

$$\frac{du^{\lambda}}{ds} + \Gamma^{\lambda}_{\mu\nu}u^{\mu}u^{\nu} = 0 \tag{A.3}$$

and a four-velocity,

$$u^{\mu} = (u^0, 0, 0, \Omega u^0) \tag{A.4}$$

and when $\lambda = 1$, the equation above reads,

$$\Gamma^1_{00}(u^0)^2 + 2\Gamma^1_{03}(u^0u^3) + \Gamma^1_{33}(u^3)^2 = \frac{1}{2e^{\mu}}u^0u^0(F_r - 2K_r\Omega - L_r\Omega^2) = 0 \tag{A.5}$$

in rigid rotation. Similarly, when $\lambda = 3$,

$$\Gamma^3_{00}(u^0)^2 + 2\Gamma^3_{03}(u^0u^3) + \Gamma^3_{33}(u^3)^2 = \frac{1}{2e^\mu}u^0u^0(F_z - 2K_z\Omega - L_z\Omega^2) = 0 \tag{A.6}$$

The two other equations are trivially satisfied. Therefore, the geodesic condition is reduced to

$$F_r - 2K_r\Omega - L_r\Omega^2 = 0 \tag{A.7}$$

$$F_z - 2K_z\Omega - L_z\Omega^2 = 0 \tag{A.8}$$

In the comoving frame, F is a constant, so $F_r = F_z = 0$, and $\Omega = 0$ does satisfy the above geodesic equation.

A.2.2 A Nondeflecting Geodesic

We have divided the total angular velocity into rigid rotation of dust particles and dragging of space-time (as can be seen in Eq. 3.42). In the comoving frame, dust particles are at rest, but space-time dragging is still present. Does an observer moving with the dust, then, know that space-time is rotating?

To see this, we invent a criterion of rotation deriving from classical intuition: there exists a geodesic of which only the x^0 and x^1 components are not zero, i.e,

$$u^\sigma = (u^0, u^1, 0, 0) \tag{A.9}$$

that has to satisfy

$$f u^0 u^0 - e^{\mu} u^1 u^1 = 0 \tag{A.10}$$

$$u^1 \frac{\partial u^0}{\partial r} + \frac{1}{r^2}(l f_r + k k_r) u^0 u^1 = 0 \tag{A.11}$$

$$u^1 \frac{\partial u^1}{\partial r} + \frac{1}{2e^{\mu}} f_r u^0 u^0 + \frac{1}{2}\mu_r u^1 u^1 = 0 \tag{A.12}$$

$$\frac{1}{2e^{\mu}} f_z u^0 u^0 - \frac{1}{2}\mu_z u^1 u^1 = 0 \tag{A.13}$$

$$\frac{1}{r^2}(f k_r - k f_r) u^0 u^1 = 0 \tag{A.14}$$

the first of the four equations is from the metric, and the rest from the geodesic equation. Consider the last equation (A.14), it is satisfied only if

$$f k_r - k f_r = 0 \quad \text{or} \quad u^0 u^1 = 0 \tag{A.15}$$

and looking at the metric (A.10), it is easily seen that neither of u^0 and u^1 can be zero, therefore it can be concluded that, the first equation in (A.15) is the condition needed for the existence of such a geodesic.

For Van Stockum's solution, no transformation of the kind $\varphi \to t + \Omega\varphi$ can bring the system to satisfy the criterion. It is intrinsically rotating.

A.3 A Nonrotating Observer on Axis

A nonrotating observer on the axis of rotation is defined, by Islam [9], as an observer along whose geodesic the unit vectors in the other two coordinate directions, i.e. $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\varphi}}$ are parallel-transported. The intrinsic angular velocity is the angular velocity with respect to this very observer, i.e., this observer must conclude an angular velocity of Ω consistent with our assumption.

First of all, a transformation to the rest frame where such an observer exist must be performed. The expression of the angular velocity in the co-moving frame (Eq. 3.43) suggests that it is very unlikely that such an observer should exist in this frame. Therefore, one alternatively considers,

$$F^{\star} = F - 2\Omega^{\star} K - \Omega^{\star 2} L \quad K^{\star} = K + \Omega^{\star} L \quad L^{\star} = L \tag{A.16}$$

in the same fashion that we transformed from the original frame into the co-moving frame.

Secondly, the four-velocity of this observer, who is at rest, can be written as

$$u^{\gamma} = (F^{\star \frac{1}{2}}, 0, 0, 0) \tag{A.17}$$

One might want to check if this is indeed a geodesic, through

$$u^{\gamma} \nabla_{\gamma} u^{\sigma} = \frac{du^{\sigma}}{ds} + \Gamma^{\sigma}_{\lambda\rho} u^{\lambda} u^{\rho} = \frac{du^{\sigma}}{ds} + \Gamma^{\sigma}_{00} u^{0} u^{0} = 0 \tag{A.18}$$

A second look at the first term

$$\frac{d}{ds} u^{\sigma} = \left(\frac{dr}{ds} \frac{d}{dr} + \frac{dz}{ds} \frac{d}{dz} \right) u^{\sigma} = \left(u^{1} \frac{d}{dr} + u^{2} \frac{d}{dz} \right) u^{\sigma} \tag{A.19}$$

reveals that it is always zero, since $u^1 = u^2 = 0$. And refer to the list of Christoffel symbols in A.1, it is obvious that only two equations are not trivial,

$$\frac{e^{-\mu}}{2} \frac{F^{\star}_{r}}{F^{\star}} = 0 \tag{A.20}$$
$$\frac{e^{-\mu}}{2} \frac{F^{\star}_{z}}{F^{\star}} = 0 \tag{A.21}$$

which holds true for a transformation in Van Stockum's solution.

Next, parallel transport of a unit vector along the geodesic of the observer (Eq. A.17) translates into two equations

$$u^{\sigma} \frac{\partial \zeta^{\nu}}{\partial x^{\sigma}} + \Gamma^{\nu}_{\lambda\gamma} u^{\lambda} \zeta^{\gamma} = 0 \tag{A.22}$$

$$\zeta^{\nu} = (0, e^{-\frac{\mu}{2}}, 0, 0) \tag{A.23}$$

and they reduce to

$$(LF_r + KK_r) + \Omega^\star(KL_r - LK_r) = 0 \tag{A.24}$$

$$(FK_r - KF_r) + \Omega^\star(FL_r - LF_r) + \Omega^{\star 2}(LK_r - KL_r) = 0 \tag{A.25}$$

Notice that the term F_r can be dropped. One might think that the condition that F is a constant function is not necessary if one moves to more general situations, but without this condition, the worldline of the nonrotating observer will not be a geodesic (Eq. A.20 and A.21). The terms are kept here just for the symmetric form that these two equations possess.

Finally, substituting in Van Stockum's solution (Eq. 3.35), one gets

$$\frac{\alpha + \Omega^\star}{r} - 2\alpha^2\Omega^\star r + \alpha^3\Omega^{\star 2}r^3|_{r=0} = 0 \tag{A.26}$$

$$r(1 - \Omega^\star \alpha r^2)|_{r=0} = 0 \tag{A.27}$$

and this requires that $\alpha = -\Omega^\star$.

For the other unit tangent vector, conditions for parallel transport are given by a modification of equation (A.24 and A.25) if we change all the $_r$ to $_z$. Since Van Stockum's solution does not depend on z, these conditions are automatically satisfied.

Therefore we can conclude that $\alpha = -\Omega^\star$.

Since it is assumed that in the original frame dust particles have an angular velocity of Ω, one therefore demands that the two transformations, through Eq. 3.7 and *A.16*, the metric goes back to its original frame, i.e.,

$$\Omega = -\Omega^\star \tag{A.28}$$

and consequently α is fixed as

$$\alpha = \Omega \tag{A.29}$$

By defining a reference, the nonrotating observer, one is able to interpret the angular velocity in a consistent way: Ω refers to the angular velocity of rotation of the dust particles with respect to this observer, in whose rest frame the dust particles have a four velocity in the form $(u^0, 0, 0, \Omega u^0)$.

A subtlety noteworthy is that, if one uses the metric in the nonrotating frame,

$$F^\star = 1 + \alpha^2 r^2 + \alpha^4 r^4 \quad K^\star = \alpha^3 r^4 \quad L^\star = r^2(1 - \alpha^2 r^2) \tag{A.30}$$

and assumes that the particles are rotating rigidly,

$$u^\sigma = (u^0, 0, 0, \Omega u^0) \tag{A.31}$$

geodesic equations are satisfied as long as,

$$\Omega = \pm\alpha \tag{A.32}$$

which suggests that there is another class of particles that could rotate with the same rate and in the same manner, but just in the opposite direction. This is like the mirror image of our original solution.

One concern about using Islam's approach to fix the parameters is that, the metric must satisfy the two conditions (Eq. A.22 and A.23). A more complicated case whereby $(F - 2\Omega^\star K - \Omega^{\star 2} L)$ has nonvanishing derivatives of r and z on the axis so that the rest observer is following a geodesic, will not be able to use this method.

A.4 Bessel Functions

A Bessel function of the first kind $y = J_m(x)$ is defined as

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - m^2)y = 0 \tag{A.33}$$

An important recurrence property is that,

$$\frac{dJ_0(x)}{dx} = J_1(x) \tag{A.34}$$

It is trivial to see that if $y(x) = J_0(kx)$, then

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} = -(k^2 x^2) y \tag{A.35}$$

The asymptotic form of J_m, when $x \ll |m^2 - \frac{1}{4}|$, can be written as,

$$J_m(x) = \sqrt{\frac{2}{\pi x}} \left(\cos \left(x - \frac{m\pi}{2} - \frac{\pi}{4} \right) \right) \tag{A.36}$$

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