

Solutions in Einstein-Maxwell-Dilaton Gravity



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Abstract

In this project, we investigated solutions in Einstein-Maxwell-Dilaton (EMD) gravity with Liouville potentials and its subset systems. We first performed a case study between two solutions found by two different authors for pure Einstein and Einstein-Dilaton gravity and showed that they were related through Kasner's second condition. We then proceeded to derive a class of general solutions for EMD gravity with the addition of Liouville couplings to the Maxwell field and cosmological constant by introducing assumptions for our metric ansatz. We showed that the resulting field equations are exactly solvable for four specific cases: 1. " $q = 0$ ", 2. " $\Lambda = 0$ ", 3. " $\alpha = \beta = 1, D = 3$ " and 4. " $\alpha\beta = 1, D = 3$ ". We found that Cases 1 and 2 are in agreement with the Cosmic Censorship Hypothesis whereas Cases 3 and 4 violate it. We then showed that our solution for Case 1 reproduces the correct curvature behavior when reduced back to the pure Einstein and Einstein-Dilaton system. We also showed that our solution for Case 2 may be expressed in a Melvin-like form, and under a specific coordinate transformation, we demonstrated that this solution reduces back to Minkowski spacetime in the limit where Maxwell fields are zero, agreeing with Melvin's solution. We also plot the effective potentials for time-like geodesics in Cases 1 and 2, and showed that varying the Liouville coupling constants and an introduced parameter, b , could remove points of equilibrium, and also change the spacetime from an attractive system to a repulsive one.

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Conventions

For the benefit of the reader, we will list some mathematical conventions used in this thesis below:

$c = 8\pi G = 1$	We will use the modified Planck units for simplicity.
$(-, +, +, +)$	Our metric signature convention.
T , or T ;	Commas and semicolons are solely for labelling purposes and will <i>not</i> denote partial or covariant derivatives. We will explicitly write out the derivative operators for clarity.
$R_{\mu\nu}$, R^{ij}	Greek indices will run across all coordinates while latin indices will <i>exclude</i> the independent coordinate variable ρ .
$f'(\rho)$	A prime symbol denotes a partial derivative with respect to ρ acting on the function, ie. $\frac{\partial}{\partial\rho}f(\rho)$.
$F^2 = F_{\mu\nu\dots}F^{\mu\nu\dots}$	The square of any tensor will be defined as such.

Chapter 1

Introduction

1.1 Motivations for the Search of Solutions in General Relativity

The search for solutions in Einstein's general relativity is by far no easy task. In general, this involves solving 10 different, coupled non-linear partial differential equations with multiple independent variables from the famous Einstein field equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}. \quad (1.1)$$

Nevertheless, many exact solutions have been found over the years using a wide spectrum of methods — examples of less exotic ones include simplifying the problem with certain assumptions, applying symmetries or choosing a suitable ansatz for the solution.

The physical interpretation of many solutions are well known, including notable ones like the Schwarzschild black hole [1], the Kerr-Newman rotating black hole [2], the Friedmann-Lemaître-Robertson-Walker (FLRW) metric [3], etc. However, there also exist many other solutions which solve Einstein's field equations but yet may not have physical interpretations attached to them.

Nonetheless, this does not imply that all of such unphysical solutions are meaningless. For example, the Anti-de Sitter/conformal field theory (AdS/CFT) correspondence is a conjecture which relates gravitational theory — the former — to quantum field theories — the latter. This was first proposed by Maldacena [4] in 1999 and is also

sometimes known as the Maldacena duality. In a nutshell, a particular theory in CFT may be described by a corresponding spacetime in AdS space. While both theories are technically different, the AdS/CFT correspondence conjectures them to be exactly equivalent. Therefore, even though black hole solutions in AdS spacetimes are generally physically unrealistic, these solutions can be linked to its corresponding theory in CFT and are useful in understanding properties of the theory. The full details of the AdS/CFT correspondence may be found in references [5] and [6].

1.2 Objectives of this Thesis

In this thesis, we will be focusing on Einstein-Maxwell-Dilaton (EMD) gravity with Liouville potentials its subset systems (ie. turning off the electromagnetic fields or scalar field). We will introduce the EMD action and our metric ansatz in Chapter 2 and also develop the mathematical tools needed to proceed with the bulk of calculations used in further chapters. Chapter 3 is a case study investigating the link between two solutions in pure Einstein gravity and Einstein-Dilaton gravity, both of which are subsets of EMD gravity.

The next few chapters will be our primary interest in this thesis where we derive an entire class of solutions using our choice of metric ansatz and a set of assumptions. Our approach was inspired by Maki's paper [7] where she uses a similar ansatz to solve for (2+1)-dimension EMD gravity. Chapter 4 presents an extension to her methods by introducing arbitrary dimensions and thereby exhausting all possible solutions attainable by this means. We also take a different approach to interpreting our coordinate system and investigate the singularities exhibited by our obtained solutions in Chapter 5. Chapter 6 will dwell deeper into two of our four obtained solutions. We will show that the limiting cases of our solutions will reproduce well-known spacetimes like the AdS black hole or the dilaton-Melvin [8][9], and original Melvin solution [10]. We will also briefly consider the geodesic structures of these spacetimes and the consequences of the coupling parameters on the shape of the effective potential.

We will then wrap up this thesis with a final conclusion in Chapter 7.

Chapter 2

The Einstein-Maxwell-Dilaton Action

In the EMD theory we are interested in, the action is given by

$$S = \frac{1}{2} \int d^D x \sqrt{-g} (R - 2\Lambda e^{2\beta\varphi} - (\nabla\varphi)^2 - e^{-2\alpha\varphi} F^2), \quad (2.1)$$

where S is the action, D is the dimensionality of the system, R is the Ricci scalar which represents the Einstein-Hilbert term, Λ is the cosmological constant, φ is the Dilaton or scalar field, F is the Maxwell 2-form (also known as the electromagnetic (EM) tensor) defined by $F = dA$ – the exterior derivative of the gauge potential A – and α and β are coupling constants.

The exponential couplings to both the cosmological constant and the electromagnetic tensor are also known as *Liouville couplings* (or potentials) and have their basis in non-critical string theories [11][12]. Setting both couplings to zero would yield back the normal Lagrangian terms in the action (for eg. we will obtain the typical Maxwell field equations when varying the EM action, and the Klein-Gordon equation [13][14] for a massless scalar field for the Dilaton action). In this thesis, we will only be interested in positive values of α and β .

While the couplings make it harder to find solutions to the field equations, they are parameters which are tunable and therefore can give rise to new classes of solutions as compared to the system with $\alpha = \beta = 0$.

2.1 The Field Equations

In this section, we will derive the field equations by applying calculus of variations to the action. Varying the action gives us

$$\begin{aligned} \delta S = & \frac{1}{2} \int d^D x \sqrt{-g} \left[\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda e^{2\beta\varphi} g_{\mu\nu} + \frac{1}{2} g_{\mu\nu} (\nabla\varphi)^2 - \nabla_\mu\varphi \nabla_\nu\varphi \right. \right. \\ & \left. \left. - 2e^{-2\alpha\varphi} F_{\nu\lambda} F_\mu^\lambda + \frac{1}{2} e^{-2\alpha\varphi} F^2 g_{\mu\nu} \right) \delta g^{\mu\nu} - 2 \left((\nabla^2\varphi) + \alpha e^{-2\alpha\varphi} F^2 - 2\Lambda\beta e^{2\beta\varphi} \right) \delta\varphi \right. \\ & \left. - 4 \left(\nabla_\sigma e^{-2\alpha\varphi} F^{\sigma\lambda} \right) \delta A_\lambda \right]. \end{aligned} \quad (2.2)$$

The full mathematical details of the above variation may be found in Appendix A. The corresponding field equations are simply found by setting the variation to zero. Therefore, each (...) must be zero such that the entire integral is zero. Thus we have three field equations corresponding to the field variables $g^{\mu\nu}$, φ and A_λ respectively:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda e^{2\beta\varphi} g_{\mu\nu} + \frac{1}{2} g_{\mu\nu} (\nabla\varphi)^2 - \nabla_\mu\varphi \nabla_\nu\varphi - 2e^{-2\alpha\varphi} F_{\nu\beta} F_\mu^\beta + \frac{1}{2} e^{-2\alpha\varphi} F^2 g_{\mu\nu} = 0, \quad (2.3)$$

$$\nabla^2\varphi + \alpha e^{-2\alpha\varphi} F^2 - 2\Lambda\beta e^{2\beta\varphi} = 0, \quad (2.4)$$

$$\nabla_\sigma (e^{-2\alpha\varphi} F^{\sigma\lambda}) = 0. \quad (2.5)$$

We may eliminate the Ricci scalar from Eq. (2.3) by taking the trace of the equation and rearranging:

$$\begin{aligned} R - \frac{D}{2} R + \Lambda e^{2\beta\varphi} D + \frac{D}{2} (\nabla\varphi)^2 - (\nabla\varphi)^2 - 2e^{-2\alpha\varphi} F^2 + \frac{D}{2} e^{-2\alpha\varphi} F^2 &= 0 \\ \Rightarrow R = \frac{2D}{D-2} \Lambda e^{2\beta\varphi} + (\nabla\varphi)^2 + \frac{D-4}{D-2} e^{-2\alpha\varphi} F^2. \end{aligned} \quad (2.6)$$

Substituting the Ricci scalar back into Eq. (2.3), we get

$$\begin{aligned} R_{\mu\nu} = & \frac{1}{2} g_{\mu\nu} \left(\frac{2D}{D-2} \Lambda e^{2\beta\varphi} + (\nabla\varphi)^2 + \frac{D-4}{D-2} e^{-2\alpha\varphi} F^2 \right) \\ & - \Lambda e^{2\beta\varphi} g_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (\nabla\varphi)^2 + \nabla_\mu\varphi \nabla_\nu\varphi + 2e^{-2\alpha\varphi} F_{\nu\beta} F_\mu^\beta - \frac{1}{2} e^{-2\alpha\varphi} F^2 g_{\mu\nu} \\ = & \frac{2\Lambda}{D-2} e^{2\beta\varphi} g_{\mu\nu} + 2e^{-2\alpha\varphi} F_{\nu\beta} F_\mu^\beta - \frac{1}{D-2} e^{-2\alpha\varphi} F^2 g_{\mu\nu} + \nabla_\mu\varphi \nabla_\nu\varphi. \end{aligned} \quad (2.7)$$

We will be referring to the following paper by Lim [15] and therefore we shall make some swaps in dummy indices to match his notation for the field equations. To sum up, we have the following set of equations for our fields:

$$R_{\mu\nu} = \frac{2\Lambda}{D-2} e^{2\beta\varphi} g_{\mu\nu} + 2e^{-2\alpha\varphi} F_{\mu\lambda} F_{\nu}^{\lambda} - \frac{1}{D-2} e^{-2\alpha\varphi} F^2 g_{\mu\nu} + \nabla_{\mu}\varphi \nabla_{\nu}\varphi, \quad (2.8a)$$

$$\nabla^2\varphi + \alpha e^{-2\alpha\varphi} F^2 - 2\Lambda\beta e^{2\beta\varphi} = 0, \quad (2.8b)$$

$$\nabla_{\sigma}(e^{-2\alpha\varphi} F^{\sigma\lambda}) = 0. \quad (2.8c)$$

2.2 Metric Ansatz

In this section, we will introduce our metric ansatz which will be used in most calculations in the following section. For simplicity, we consider a Weyl-like, diagonal metric carrying a Lorentzian signature with $(D-1)$ Killing vectors (ie. all metric coefficients are only a function of one coordinate, ρ):

$$ds^2 = -e^{2F_0(\rho)} dt^2 + e^{2H(\rho)} d\rho^2 + \sum_{i=1}^n e^{2F_i(\rho)} dx_i^2. \quad (2.9)$$

There are three advantages to using an ansatz of the form above:

1. Computation of differential geometry quantities such as the Christoffel symbols, Ricci tensor components etc., are greatly simplified since the derivatives of exponential functions are straightforward.
2. The metric components are entirely arbitrary which provides us with a more general spacetime ansatz. Therefore we have a higher chance of obtaining new solutions different from the many already-known solutions in EMD gravity, many of which has the general form

$$ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + h(r) d\Omega_{D-2}^2. \quad (2.10)$$

3. The Liouville couplings in the action foreshadows the appearance of the Liouville differential equation which will be introduced in Chapter 4. Having exponential metric components allows us to obtain the desired form of the differential equation, simplifying the solving process.

We will proceed to derive the differential geometry quantities corresponding to our metric ansatz in the sections following.

2.2.1 Christoffel Symbols

Since we are considering a torsionless manifold, the Christoffel symbols are symmetric in the bottom two indices and are given by

$$\Gamma_{\mu\nu}^{\kappa} = \frac{1}{2}g^{\kappa\lambda}(\partial_{\mu}g_{\lambda\nu} + \partial_{\nu}g_{\lambda\mu} - \partial_{\lambda}g_{\mu\nu}). \quad (2.11)$$

The full details of our calculation are available in Appendix B. Here, we will simply list the non-zero Christoffel symbols:

$$\Gamma_{tt}^{\rho} = e^{2F_0-2H}F_0', \quad (2.12a)$$

$$\Gamma_{ii}^{\rho} = -e^{2F_i-2H}F_i', \quad (2.12b)$$

$$\Gamma_{\rho\rho}^{\rho} = H', \quad (2.12c)$$

$$\Gamma_{t\rho}^t = \Gamma_{\rho t}^t = F_0', \quad (2.12d)$$

$$\Gamma_{i\rho}^i = \Gamma_{\rho i}^i = F_i'. \quad (2.12e)$$

2.2.2 Ricci Tensor Components and Ricci Scalar

The Ricci tensor is given by

$$R_{\sigma\nu} = \partial_{\mu}\Gamma_{\nu\sigma}^{\mu} - \partial_{\nu}\Gamma_{\mu\sigma}^{\mu} + \Gamma_{\mu\lambda}^{\mu}\Gamma_{\nu\sigma}^{\lambda} - \Gamma_{\nu\lambda}^{\mu}\Gamma_{\mu\sigma}^{\lambda}. \quad (2.13)$$

For our metric, the Ricci components are then

$$R_{tt} = e^{2F_0-2H} \left(F_0'' - H'F_0' + F_0' \sum_{i=0}^n F_i' \right), \quad (2.14a)$$

$$R_{ii} = -e^{2F_i-2H} \left(F_i'' - H'F_i' + F_i' \sum_{j=0}^n F_j' \right), \quad (2.14b)$$

$$R_{\rho\rho} = \sum_{i=0}^n (H'F_i' - F_i'' - F_i'^2). \quad (2.14c)$$

The details of the above calculations may be found in Appendix C. We can then obtain the Ricci scalar from

$$\begin{aligned}
R &= g^{\mu\nu} R_{\mu\nu} \\
&= g^{tt} R_{tt} + g^{\rho\rho} R_{\rho\rho} + \sum_{i=1}^n g^{ii} R_{ii} \\
&= -e^{-2H} \left[F_0'' - H' F_0' + F_0' \sum_{i=0}^n F_i' \right] + e^{-2H} \sum_{i=0}^n (H' F_i' - F_i'' - F_i'^2) \\
&\quad - \sum_{i=1}^n e^{-2H} \left[F_i'' - H' F_i' + F_i' \sum_{j=0}^n F_j' \right] \\
&= e^{-2H} \sum_{i=0}^n \left(2H' F_i' - 2F_i'' - F_i'^2 - F_i' \sum_{j=0}^n F_j' \right). \tag{2.15}
\end{aligned}$$

Chapter 3

Case Study: Pure Einstein vs Einstein-Dilaton Gravity

In this chapter, we will be analysing the metric solutions from Ren [16] which describes pure Einstein gravity with a negative cosmological constant, and from Lim [15] which describes Einstein-Dilaton gravity. Even though the action in which both authors were considering are different, their obtained metric solutions bear resemblance to each other. Hence, we are interested in finding the link between them and investigating whether there is any physical significance to it.

3.1 The Field Equations and Metric Solutions

3.1.1 Field Equations

The actions considered by Ren and Lim are given as

$$S_{\text{Ren}} = \frac{1}{2} \int d^D x \sqrt{-g} \left(R + \frac{n(n+1)}{l^2} \right), \quad (3.1)$$

$$S_{\text{Lim}} = \frac{1}{2} \int d^D x \sqrt{-g} (R - 2\Lambda - (\nabla\varphi)^2), \quad (3.2)$$

where $n = D - 2$ (we will be freely interchanging between n and D for computational convenience) and l is the Anti-de Sitter radius. Ren's action describes the AdS_{n+2} black hole while Lim's action describes the AdS_{n+2} naked singularity. We are free to

then define

$$-2\Lambda := \frac{n(n+1)}{l^2}. \quad (3.3)$$

Thus we can see that the system Lim is considering is simply Ren's with the inclusion of a scalar field. Since both actions are a simplification of the general EMD action we have introduced in Chapter 2, we simply need to turn off the relevant quantities (ie. electromagnetic fields and coupling constants) from Eq. (2.8) to obtain the associated field equations for the above systems:

$$R_{\mu\nu, \text{Ren}} = -\frac{n+1}{l^2} g_{\mu\nu}, \quad (3.4)$$

$$R_{\mu\nu, \text{Lim}} = -\frac{n+1}{l^2} g_{\mu\nu} + \nabla_\mu \varphi \nabla_\nu \varphi, \quad (3.5)$$

$$\nabla^2 \varphi = 0. \quad (3.6)$$

3.1.2 Metric Solutions

We also present their metric solutions in their original notations. We will be placing a bar ($\bar{}$) notation on Lim's coordinates to remind us that both authors' coordinate choices are independent of each other and are not related. Ren's metric is given as

$$ds_{\text{Ren}}^2 = \frac{l^2}{r^2} \left(-f^{p_t} dt^2 + \frac{dr^2}{f} + \sum_{i=1}^n f^{p_i} dx_i^2 \right), \quad f = 1 - \left(\frac{r}{r_0} \right)^{n+1}, \quad (3.7)$$

where the exponents p_t and p_i are constants which obey Kasner's conditions of which we will expound on in the next section, and r_0 is some arbitrary reference radius. Lim's metric is given as

$$ds_{\text{Lim}}^2 = -\frac{\bar{r}^2}{l^2} f_L^{\frac{\nu(D-2)+1}{D-1}} d\bar{t}^2 + \frac{l^2}{\bar{r}^2} \frac{d\bar{r}^2}{f_L} + \sum_{i=1}^n \frac{\bar{r}^2}{l^2} f_L^{\frac{1-\nu}{D-1}} d\bar{x}_i^2, \quad f_L = 1 - \frac{\mu}{\bar{r}^{n+1}}, \quad (3.8)$$

where ν is a tunable parameter and μ is some arbitrary reference radius. As stated in his paper, the above metric is the same as the one found by Saenz and Mertinez [17], but we will refer to it in this thesis as Lim's metric for simplicity. Although there are some obvious similarities between Eq. (3.7) and Eq. (3.8), they are not exactly

in the same form. To cast them into similar expressions, we introduce the following change of variables and rescaling of coordinates:

$$\begin{aligned} \bar{r} &= \frac{1}{r}, & \bar{t} &= l^2 t, & \bar{x}_i &= l^2 x_i, \\ \Rightarrow d\bar{r}^2 &= \frac{1}{r^2} dr^2, & d\bar{t}^2 &= l^4 dt^2, & d\bar{x}_i^2 &= l^4 dx_i^2, \\ & & \mu &= \frac{1}{r_0^{n+1}}. \end{aligned} \quad (3.9)$$

Lim's metric with the new coordinates is then

$$ds_{\text{Lim}}^2 = \frac{l^2}{r^2} \left(-f^{\frac{\nu(D-2)+1}{D-1}} dt^2 + \frac{dr^2}{f} + \sum_{i=1}^n f^{\frac{1-\nu}{D-1}} dx_i^2 \right). \quad (3.10)$$

Now it is obvious that both metric solutions in Eq. (3.7) and Eq. (3.10) are almost identical to each other apart from the exponents of the function f . Lim's exponents are parameterised by ν whilst Ren's is still somewhat general.

3.2 Kasner's Conditions

Ren's metric in Eq. (3.7) is also known as the Kasner metric. It was developed and named after the American mathematician Edward Kasner in 1921 [18]. In his paper, Kasner showed that for the metric to be a vacuum solution for the Einstein field equation, p_t ($:= p_0$) and p_i (also known as the Kasner exponents) must satisfy two conditions

$$\text{First Condition: } \sum_{i=0}^n p_i = 1, \quad (3.11)$$

$$\text{Second Condition: } \sum_{i=0}^n p_i^2 = 1. \quad (3.12)$$

The first intuitive thing to do is to check whether Lim's exponents also satisfy Kasner's conditions or not:

$$\begin{aligned} \text{First Condition: } \sum_{i=0}^n p_i &= \frac{\nu n + 1}{n + 1} + \sum_{i=1}^n \frac{1 - \nu}{n + 1} \\ &= \frac{\nu n + 1}{n + 1} + n \frac{1 - \nu}{n + 1} \end{aligned}$$

$$= 1, \tag{3.13}$$

$$\begin{aligned} \text{Second Condition: } \sum_{i=0}^n p_i^2 &= \left(\frac{\nu n + 1}{n + 1}\right)^2 + \sum_{i=1}^n \left(\frac{1 - \nu}{n + 1}\right)^2 \\ &= \frac{\nu^2 n^2 + 1 + 2\nu n}{(n + 1)^2} + n \frac{1 - 2\nu + \nu^2}{(n + 1)^2} \\ &= \frac{(\nu^2 n + 1)(n + 1)}{(n + 1)^2} \\ &= 1 - \frac{n(\nu^2 - 1)}{n + 1}. \end{aligned} \tag{3.14}$$

Since n cannot be 0 as the spacetime of interest is at least 3-dimensional, therefore, it is clear that Kasner's second condition for Lim's metric does not hold except for the case when $\nu^2 = 1$. The significance of this special case will be made clear in the following sections.

3.3 The Scalar Field and the R_{rr} Term

For Lim's case, he also has a solution for the scalar field given as

$$\varphi(r) = \frac{1}{2} \sqrt{\frac{n(1 - \nu^2)}{n + 1}} \ln f. \tag{3.15}$$

Since the scalar field is only dependent on the r -coordinate, the field equations from Eq. (3.4) and Eq. (3.5) for the R_{tt} and R_{ii} components will be exactly the same in both Lim's and Ren's system since there is no contribution from the scalar field. However, the R_{rr} component will be of particular interest as we would like to see the effect of having an additional scalar field in the field equation.

We have explicitly verified that both authors' given metric are true solutions to the field equations and the mathematical details are left to Appendix D. However, we will show the workings for the R_{rr} component here in detail to investigate the link between both authors' solutions.

3.3.1 Field Equation for the R_{rr} Term

In Chapter 2, we have developed the mathematical tools to calculate the Ricci tensor components for our metric ansatz. We may apply our results to calculate the R_{rr} component for Lim's metric by expressing the metric in the exponential form and obtaining the ansatz exponents.

It turns out that we may simplify the calculations by first performing a coordinate transformation of

$$\begin{aligned}\rho &= -\ln r, \\ \Rightarrow d\rho^2 &= \left(-\frac{1}{r}dr\right)^2 = \frac{1}{r^2}dr^2, \\ \Rightarrow \frac{1}{r^2} &= e^{2\rho}.\end{aligned}\tag{3.16}$$

Lim's metric from Eq. (3.10) is then

$$ds_{\text{Lim}}^2 = -l^2 e^{2\rho} f^{\frac{\nu n+1}{n+1}} dt^2 + \frac{l^2}{f} d\rho^2 + l^2 \sum_{i=1}^n e^{2\rho} f^{\frac{1-\nu}{D-1}} dx_i^2.\tag{3.17}$$

We may now compare coefficients and obtain the ansatz exponents accordingly as defined in Chapter 2

$$\begin{aligned}e^{2F_0} &= l^2 e^{2\rho} f^{\frac{\nu n+1}{n+1}}, \\ \Rightarrow F_0 &= \rho + \frac{1}{2} \frac{\nu n+1}{n+1} \ln f + \ln l,\end{aligned}\tag{3.18a}$$

$$\begin{aligned}e^{2H} &= \frac{l^2}{f}, \\ \Rightarrow H &= -\frac{1}{2} \ln f + \ln l,\end{aligned}\tag{3.18b}$$

$$\begin{aligned}e^{2F_i} &= l^2 e^{2\rho} f^{\frac{1-\nu}{n+1}}, \\ \Rightarrow F_i &= \rho + \frac{1}{2} \frac{1-\nu}{n+1} \ln f + \ln l.\end{aligned}\tag{3.18c}$$

Note that the R_{rr} term will now be referred to $R_{\rho\rho}$ due to the coordinate change. As in Eq. (2.14c), the $R_{\rho\rho}$ term is given by

$$R_{\rho\rho} = \sum_{i=0}^n (H' F_i' - F_i'' - F_i'^2)$$

Substituting in accordingly and simplifying,

$$\begin{aligned}
R_{\rho\rho} &= \underbrace{-\frac{1}{2} \frac{\nu n + 1}{n + 1} \frac{f'' f - f'^2}{f^2} - \frac{f'}{2f} \left(1 + \frac{1}{2} \frac{\nu n + 1}{n + 1} \frac{f'}{f}\right) - \left(1 + \frac{1}{2} \frac{\nu n + 1}{n + 1} \frac{f'}{f}\right)^2}_{i=0 \text{ term}} \\
&\quad + \sum_{i=1}^n \left[-\frac{f'}{2f} \left(1 + \frac{1}{2} \frac{1 - \nu}{n + 1} \frac{f'}{f}\right) - \frac{1}{2} \frac{1 - \nu}{n + 1} \frac{f'' f - f'^2}{f^2} - \left(1 + \frac{1}{2} \frac{1 - \nu}{n + 1} \frac{f'}{f}\right)^2 \right] \\
&= -(n + 1) - \frac{f'}{f} \left(\frac{n + 1}{2} + \frac{\nu n + 1}{n + 1} + n \frac{1 - \nu}{n + 1} \right) - \frac{f''}{f} \left(\frac{n}{2} \frac{1 - \nu}{n + 1} + \frac{1}{2} \frac{\nu n + 1}{n + 1} \right) \\
&\quad + \frac{f'^2}{4f^2(n + 1)} \left(n(1 - \nu) - n \frac{1 + \nu^2 - 2\nu}{n + 1} - \frac{\nu^2 n^2 + 1 + 2\nu n}{n + 1} + \nu n + 1 \right) \\
&= -(n + 1) - \frac{f'}{f} \left(\frac{3 + n}{2} \right) - \frac{f''}{f} \left(\frac{1}{2} \right) + \frac{f'^2}{4f^2(n + 1)} (n(1 - \nu^2)) \\
&= -\frac{n + 1}{l^2} \frac{f'}{f} + \frac{1}{4} \frac{n(1 - \nu^2)}{n + 1} \frac{f'^2}{f^2}. \tag{3.19}
\end{aligned}$$

At this juncture, we have the first term in Eq. (3.19) equal to $-\frac{n+1}{l^2} g_{\rho\rho}$ as desired. Therefore the second term is naturally equivalent to the covariant derivative term in Eq. (3.5). As a check,

$$\begin{aligned}
\nabla_{\rho}\varphi\nabla_{\rho}\varphi &= (\partial_{\rho}\varphi)^2 \\
&= \left(\frac{1}{2} \sqrt{\frac{n(1 - \nu^2)}{n + 1}} \frac{f'}{f} \right)^2 \\
&= \frac{1}{4} \frac{n(1 - \nu^2)}{n + 1} \frac{f'^2}{f^2}, \tag{3.20}
\end{aligned}$$

where the equal sign in the first step comes from the fact that φ is a scalar field. Note that again in the special case of $\nu^2 = 1$, the above term goes to zero which implies that the field equations of Lim becomes identical to Ren's.

3.4 The Link Between Ren's and Lim's Solutions

A recurring theme which has been appearing in the above sections is the special case where $\nu^2 = 1$. We have seen that when this occurs, the exponents in Lim's metric satisfy Kasner's second condition. At the same time, this corresponds to the scalar field defined in Eq. (3.15) going to zero and the field equations of Lim and Ren becoming identical.

Therefore, enforcing Kasner's two conditions to hold for Lim's metric is akin to the act of switching off the scalar field and thereby reducing the system to pure Einstein gravity. The beauty in this discovery lies in the fact Kasner's conditions are essentially pure mathematical constructs, but with this case study we can see its manifestation in a physical quantity like the scalar field.

It is known that the introduction of matter fields can relax Kasner's second condition for AdS spacetimes as pointed out by Banerjee et al. [19]. In fact, the paper mentions that to do this, one would need to introduce a scalar field of the form $\Phi = \lambda \ln f$ where λ is defined by a modified Kasner's second condition given as

$$\sum_{i=0}^n p_i^2 = 1 - \frac{\lambda^2}{2} \quad (3.21)$$

This is in slight *disagreement* with what we have found above. Taking $\lambda = \frac{1}{2} \sqrt{\frac{n(1-\nu^2)}{n+1}}$ from Lim's scalar field solution in equation 3.15, we instead obtain

$$\sum_{i=0}^n p_i^2 = 1 - 4\lambda^2 \quad (3.22)$$

The difference could be resolved by allowing a rescaling on λ or Φ but their paper is not very clear at how they arrived at their results. On the contrary, we remain confident in our findings as we have demonstrated clearly in this chapter how we arrived at our conclusion. Note that we do not claim Eq. (3.22) to be generally true.

Chapter 4

Deriving a New Class of Solutions

In this chapter, we will show how we derive an entire class of solutions in EMD gravity. While we have already obtained the field equations for EMD gravity in Eq. (2.8) and also introduced our metric ansatz in Eq. (2.9) in Chapter 2, we need to make some assumptions for the scalar and EM fields first.

Following the motivations of our metric, we assume that all fields are also only dependent on the ρ -coordinate. We will further assume the gauge potential has a non-zero time component and zero spatial components (ie. electrically charged):

$$\varphi = \varphi(\rho), \tag{4.1}$$

$$A_\mu = (A_0(\rho), 0). \tag{4.2}$$

The reason for this is that Maki [7] has already shown that for this particular ansatz in EMD gravity, no static, rotationally symmetric dyonic solution exists. Therefore, there can only be either a magnetic or electric solution. Furthermore, reference [20] also shows that one may obtain the magnetic solution by simply flipping the sign of the coupling constant $\alpha \rightarrow -\alpha$.

4.1 Modified Field Equations

Using the above-mentioned assumptions, our field equations may now be written as

$$A' = qe^{2\alpha\varphi} e^{H+F_0-\sum_{i=1}^n F_i}, \tag{4.3a}$$

$$\varphi'' = \left(H' - \sum_{i=0}^n F'_i \right) \varphi' + 2\alpha q^2 e^{2\alpha\varphi+2H-2\sum_{i=1}^n F_i} + 2\Lambda\beta e^{2\beta\varphi+2H}, \quad (4.3b)$$

$$F_0'' = F_0' \left(H' - \sum_{i=0}^n F'_i \right) - \frac{2\Lambda}{D-2} e^{2\beta\varphi+2H} + 2q^2 \left(\frac{D-3}{D-2} \right) e^{2\alpha\varphi+2H-2\sum_{i=1}^n F_i}, \quad (4.3c)$$

$$F_{i \neq 0}'' = F_{i \neq 0}' \left(H' - \sum_{j=0}^n F'_j \right) - \frac{2\Lambda}{D-2} e^{2\beta\varphi+2H} - \frac{2q^2}{D-2} e^{2\alpha\varphi+2H-2\sum_{i=1}^n F_i}, \quad (4.3d)$$

$$\sum_{i=0}^n (H' F'_i - F_i'' - F_i'^2) = \frac{2\Lambda}{D-2} e^{2\beta\varphi+2H} - 2q^2 \left(\frac{D-3}{D-2} \right) e^{2\alpha\varphi+2H-2\sum_{i=1}^n F_i} + \varphi'^2. \quad (4.3e)$$

The newly introduced constant, q , is some electric charge parameter which came from the direct integration of the EM field equation. The derivation of the above results are shown in detail in Appendix E. Eq. (4.3a) corresponds to the EM field, Eq. (4.3b) corresponds to the scalar field and Eq. (4.3c), Eq. (4.3d) and Eq. (4.3e) comes from the Einstein field equation.

We may proceed with a simplification by first summing all $i \neq 0$ components from Eq. (4.3d), noting that $n = D - 2$:

$$2\Lambda e^{2\beta\varphi+2H} + 2q^2 e^{2\alpha\varphi+2H-2\sum_{i=1}^n F_i} = \sum_{i=1}^n \left(H' F'_i - F_i'' - F_i' \sum_{j=0}^n F'_j \right). \quad (4.4)$$

We then add together Eq. (4.3c) and Eq. (4.4) and get

$$2\Lambda \left(\frac{D-1}{D-2} \right) e^{2\beta\varphi+2H} + 2q^2 e^{2\alpha\varphi+2H-2\sum_{i=1}^n F_i} \left(\frac{1}{D-2} \right) = \sum_{i=0}^n \left(H' F'_i - F_i'' - F_i' \sum_{j=0}^n F'_j \right). \quad (4.5)$$

We then subtract Eq. (4.3e) from Eq. (4.5) to obtain

$$\sum_{ij, i \neq j}^n F'_i F'_j = -2\Lambda e^{2\beta\varphi+2H} - 2q^2 e^{2\alpha\varphi+2H-2\sum_{i=1}^n F_i} + \varphi'^2. \quad (4.6)$$

This will replace Eq. (4.3e) and will serve as a constraint equation. We will explain more about this further on in the chapter.

4.2 Assumptions for Metric Exponents

At this point, it is non-trivial to solve the current set of coupled, second-order differential equations and therefore we need to introduce certain assumptions to simplify the problem.

4.2.1 First Assumption

Assumption 1 All $F_{i \neq 0}$'s are the same, ie. $F_{i \neq 0} = F_{j \neq 0}, \forall i \neq j$.

This assumption removes the ambiguity of arbitrary functions/constants appearing after integration of the differential equations with F_i 's. Physically, this may be interpreted as having an isotropic, flat space in the background of our spacetime.

With this, we may simplify the double summation in Eq. (4.6)

$$\begin{aligned}
 \sum_{ij, i \neq j}^n F'_i F'_j &= F'_0 \underbrace{(F'_1 + F'_2 + \dots + F'_n)}_{n \text{ terms}} \\
 &+ F'_1 \underbrace{(F'_0 + F'_2 + \dots + F'_n)}_{n \text{ terms}} \\
 &\vdots \\
 &+ F'_n \underbrace{(F'_0 + F'_1 + \dots + F'_{n-1})}_{n \text{ terms}} \\
 &= 2(D-2)F'_0 F'_{i \neq 0} + (D-2)(D-3)F'^2_{i \neq 0}. \tag{4.7}
 \end{aligned}$$

Thus, Eq. (4.6) may now be rewritten as

$$2(D-2)F'_0 F'_{i \neq 0} + (D-2)(D-3)F'^2_{i \neq 0} = -2\Lambda e^{2\beta\varphi+2H} - 2q^2 e^{2\alpha\varphi+2F_0} + \varphi'^2. \tag{4.8}$$

4.2.2 Second Assumption

Assumption 2 The exponent of the $g_{\rho\rho}$ component is the sum of the g_{tt} and g_{ii} components, ie. $H = \sum_{i=0}^n F_i$.

The motivation behind this is because of the factor $(H' - \sum_{i=0}^n F'_i)$ appearing three times in our coupled differential equations. The assumption will greatly simplify the equations since it kills off the first order derivative terms in Eq. (4.3b), Eq. (4.3c) and Eq. (4.3d). This assumption also implies that our metric ansatz becomes

$$ds^2 = -e^{2F_0} dt^2 + e^{\sum_{i=0}^n 2F_i} d\rho^2 + \sum_{i=1}^n e^{2F_i} dx_i^2, \quad (4.9)$$

$$\Rightarrow g_{\rho\rho} = - \prod_{i=0}^n g_{ii}. \quad (4.10)$$

Maki [7], in fact, began her paper with the above metric ansatz without stating any motivations. Through our workings, it now becomes clear why such a choice of metric ansatz is desirable. One may also view this as an analagous Kasner condition for our set of solutions.

4.3 Final Simplified Field Equations

With these assumptions, the field equations are further simplified to become

$$A' = qe^{2\alpha\varphi+2F_0}, \quad (4.11a)$$

$$\varphi'' = 2\alpha q^2 e^{2\alpha\varphi+2F_0} + 2\Lambda\beta e^{2\beta\varphi+2\sum_{i=0}^n F_i}, \quad (4.11b)$$

$$F_0'' = -\frac{2\Lambda}{D-2} e^{2\beta\varphi+2\sum_{i=0}^n F_i} + 2q^2 \left(\frac{D-3}{D-2} \right) e^{2\alpha\varphi+2F_0}, \quad (4.11c)$$

$$F_{i\neq 0}'' = -\frac{2\Lambda}{D-2} e^{2\beta\varphi+2\sum_{i=0}^n F_i} - \frac{2q^2}{D-2} e^{2\alpha\varphi+2F_0}, \quad (4.11d)$$

$$2(D-2)F_0'F_{i\neq 0}' + (D-2)(D-3)F_{i\neq 0}'^2 = -2\Lambda e^{2\beta\varphi+2\sum_{i=0}^n F_i} - 2q^2 e^{2\alpha\varphi+2F_0} + \varphi'^2. \quad (4.11e)$$

The following is an outline of the steps we will take to obtain solutions from these equations:

1. A closer look at Eq. (4.11b), Eq. (4.11c) and Eq. (4.11d) will reveal that these three equations are coupled only to each other. Therefore, we will begin by decoupling these three equations, and solving for F_0 , $F_{i\neq 0}$ and φ .
2. Recall that for a sourceless system, the physical electric field is simply $E = -\nabla\phi = -A'$. Thus, having the solutions for F_0 and φ , Eq. (4.11a) immediately

gives us the electric field. Performing a simple integration would then return the gauge potential A .

- Eq. (4.11e) will serve as a constraint for any integration constants appearing in the above solutions.

4.4 Solving the Simplified Field Equations

4.4.1 Decoupling the Coupled Differential Equations

Following our first step, we may decouple the mentioned three equations by introducing the following change of variables:

$$\begin{aligned}\phi_1 &= \beta\varphi + \sum_{i=0}^n F_i, \\ \phi_2 &= \alpha\varphi + F_0, \\ \phi_3 &= \alpha\varphi + \sum_{i=1}^n F_i.\end{aligned}\tag{4.12}$$

Substituting in Eq. (4.11b), Eq. (4.11c) and Eq. (4.11d) to the above, we obtain a new set of coupled equations:

$$\begin{aligned}\phi_1'' &= \beta\varphi'' + F_0'' + (D-2)F_{i\neq 0}'' \\ &= 2\alpha\beta q^2 e^{2\phi_2} + 2\Lambda\beta^2 e^{2\phi_1} - \frac{2\Lambda}{D-2} e^{2\phi_1} + 2q^2 \left(\frac{D-3}{D-2}\right) e^{2\phi_2} - 2\Lambda e^{2\phi_1} - 2q^2 e^{2\phi_2} \\ &= 2\Lambda \left(\beta^2 - \frac{D-1}{D-2}\right) e^{2\phi_1} - 2q^2 \left(\frac{1}{D-2} - \alpha\beta\right) e^{2\phi_2},\end{aligned}\tag{4.13a}$$

$$\begin{aligned}\phi_2'' &= \alpha\varphi'' + F_0'' \\ &= 2\alpha^2 q^2 e^{2\phi_2} + 2\Lambda\alpha\beta e^{2\phi_1} - \frac{2\Lambda}{D-2} e^{2\phi_1} + 2q^2 \left(\frac{D-3}{D-2}\right) e^{2\phi_2} \\ &= 2\Lambda \left(\alpha\beta - \frac{1}{D-2}\right) e^{2\phi_1} + 2q^2 \left(\frac{D-3}{D-2} + \alpha^2\right) e^{2\phi_2},\end{aligned}\tag{4.13b}$$

$$\begin{aligned}\phi_3'' &= \alpha\varphi'' + (D-2)F_{i\neq 0}'' \\ &= 2\alpha^2 q^2 e^{2\phi_2} + 2\Lambda\alpha\beta e^{2\phi_1} - 2\Lambda e^{2\phi_1} - 2q^2 e^{2\phi_2} \\ &= 2\Lambda(\alpha\beta - 1) e^{2\phi_1} - 2q^2(1 - \alpha^2) e^{2\phi_2}.\end{aligned}\tag{4.13c}$$

While these are still coupled, we free to set particular values for q , Λ , α and β such that we decouple the above set of equations and obtain differential equations of the form

$$\phi''(\rho) = ke^{2\phi(\rho)}, \quad (4.14)$$

where k is either a positive or negative constant. This differential equation is also known as the *Liouville differential equation* and has several forms for its solution. For this thesis, we will use the following solution:

$$\text{Positive } k: \phi_+(\rho) = -\ln\left(\frac{\sqrt{k}}{b} \sinh(b\rho)\right), \quad (4.15a)$$

$$\text{Negative } k: \phi_-(\rho) = -\ln\left(\frac{\sqrt{-k}}{b} \cosh(b\rho)\right), \quad (4.15b)$$

where b is some arbitrary integration constant. The above result is derived in Appendix F. In total, there are only four cases where Eq. (4.13) may be decoupled:

1. $q = 0$
2. $\Lambda = 0$
3. $\alpha = \beta = 1, D = 3$
4. $\alpha\beta = 1, D = 3$

Note that in Maki's paper [7], she has pointed out these exact four cases in (2+1)-dimensions as well and made a note in her concluding paragraph that she is interested in finding more solutions in higher dimensions. She has however not made any publications since with regards to that statement, and therefore we show here first that even after expanding to arbitrary dimensions, one may not find any other cases for which the coupled differential equations are solvable using this particular method.

Maki also chose to focus only on Cases 3 and 4, but we will show in Chapter 6 that Cases 1 and 2 do yield interesting results in higher dimensions. We will leave further discussion in the following chapters and begin solving each case now.

4.4.2 Our Solutions

We will explicitly work out Case 1: ($q = 0$) to show our process of obtaining the final solutions, and then simply list out the solutions for the other cases for brevity sake. The full details for the other cases may be found in Appendix G.

Case 1: $q = 0$

In this case, we are turning off the EM field, and therefore this system describes one with a massless scalar field and a cosmological constant with the Liouville coupling. Eq. (4.13) is then simplified to

$$\phi_1'' = 2\Lambda \left(\beta^2 - \frac{D-1}{D-2} \right) e^{2\phi_1}, \quad (4.16a)$$

$$\phi_2'' = 2\Lambda \left(\alpha\beta - \frac{1}{D-2} \right) e^{2\phi_1} = \frac{\left(\alpha\beta - \frac{1}{D-2} \right)}{\left(\beta^2 - \frac{D-1}{D-2} \right)} \phi_1'', \quad (4.16b)$$

$$\phi_3'' = 2\Lambda (\alpha\beta - 1) e^{2\phi_1} = \frac{(\alpha\beta - 1)}{\left(\beta^2 - \frac{D-1}{D-2} \right)} \phi_1''. \quad (4.16c)$$

Because the values of β , D and Λ are not specified, Eq. (4.16a) may have two different solutions according to Eq. (4.15a) depending on the overall sign of the prefactor. We shall assume first that $2\Lambda \left(\beta^2 - \frac{D-1}{D-2} \right) < 0$. Therefore, the solutions are

$$\phi_1 = -\ln \left(\frac{1}{b} \sqrt{2\Lambda \left(\frac{D-1}{D-2} - \beta^2 \right) \cosh(b\rho)} \right), \quad (4.17a)$$

$$\phi_2 = \frac{\left(\alpha\beta - \frac{1}{D-2} \right)}{\left(\beta^2 - \frac{D-1}{D-2} \right)} \phi_1 + c_2\rho + c_3, \quad (4.17b)$$

$$\phi_3 = \frac{(\alpha\beta - 1)}{\left(\beta^2 - \frac{D-1}{D-2} \right)} \phi_1 + d_2\rho + d_3, \quad (4.17c)$$

where c_2 , c_3 , d_2 and d_3 are arbitrary integration constants. Rearranging the change of variables in Eq. (4.12), the original variables may be obtained from

$$\varphi = \frac{\phi_2 + \phi_3 - \phi_1}{2\alpha - \beta}, \quad (4.18a)$$

$$F_{i \neq 0} = \frac{\phi_3 - \alpha\varphi}{D-2}, \quad (4.18b)$$

$$F_0 = \phi_2 - \alpha\varphi. \quad (4.18c)$$

The final solutions after simplification are then:

$$\varphi = \frac{\beta}{\left(\beta^2 - \frac{D-1}{D-2}\right)} \phi_1 + \frac{c_2 + d_2}{2\alpha - \beta} \rho + \frac{c_3 + d_3}{2\alpha - \beta}, \quad (4.19a)$$

$$F_{i \neq 0} = -\frac{1}{D-2} \frac{1}{\left(\beta^2 - \frac{D-1}{D-2}\right)} \phi_1 + \frac{1}{D-2} \frac{\alpha(d_2 - c_2) - \beta d_2}{2\alpha - \beta} \rho + \frac{1}{D-2} \frac{\alpha(d_3 - c_3) - \beta d_3}{2\alpha - \beta}, \quad (4.19b)$$

$$F_0 = -\frac{1}{D-2} \frac{1}{\left(\beta^2 - \frac{D-1}{D-2}\right)} \phi_1 + \frac{\alpha(c_2 - d_2) - \beta c_2}{2\alpha - \beta} \rho + \frac{\alpha(c_3 - d_3) - \beta c_3}{2\alpha - \beta}. \quad (4.19c)$$

Substituting these into Eq. (4.11e), we get the constraint equation for the arbitrary constants:

$$\begin{aligned} c_2^2 \left[\alpha^2 - (\alpha^2 + 1) D + 2\alpha\beta(D-2) + 2 \right] + 2c_2 d_2 \left[-\alpha^2 + \beta(\alpha - 2\beta) + D(\alpha^2 - \alpha\beta + \beta^2 - 1) + 2 \right] \\ + d_2^2 \left[\alpha^2 + 2\alpha\beta - (\alpha^2 + 1) D + \beta^2(D-3) + 2 \right] = \frac{b^2(D-2)(D-2)(\beta - 2\alpha)^2}{\beta^2(D-2) - D + 1} \end{aligned} \quad (4.20)$$

We can set c_3 and d_3 to 0, which can also be interpreted as a rescaling of the coordinates. Dropping the subscripts on $c_2 \rightarrow c$ and $d_2 \rightarrow d$, the final metric solution is reconstructed to be

$$\begin{aligned} ds^2 = -e^{2\left(\frac{\alpha(c-d)-\beta c}{2\alpha-\beta}\right)\rho} G(\rho)^{\frac{1}{D-2} \frac{1}{\left(\beta^2 - \frac{D-1}{D-2}\right)}} dt^2 + e^{2(c-d)\rho} G(\rho)^{\frac{D-1}{D-2} \frac{1}{\left(\beta^2 - \frac{D-1}{D-2}\right)}} d\rho^2 \\ + \sum_{i=1}^n e^{-\frac{2}{D-2} \left(\frac{\alpha(d-c)-\beta d}{2\alpha-\beta}\right)\rho} G(\rho)^{\frac{1}{D-2} \frac{1}{\left(\beta^2 - \frac{D-1}{D-2}\right)}} dx_i^2, \end{aligned} \quad (4.21)$$

with

$$G(\rho) = \begin{cases} \frac{2\Lambda}{b^2} \left(\frac{D-1}{D-2} - \beta^2\right) \cosh^2(b\rho), & \text{for } 2\Lambda \left(\beta^2 - \frac{D-1}{D-2}\right) < 0, \\ \frac{2\Lambda}{b^2} \left(\beta^2 - \frac{D-1}{D-2}\right) \sinh^2(b\rho), & \text{for } 2\Lambda \left(\beta^2 - \frac{D-1}{D-2}\right) > 0. \end{cases} \quad (4.22)$$

The corresponding scalar field is then

$$\varphi = -\frac{\beta}{2\left(\beta^2 - \frac{D-1}{D-2}\right)} \ln G(\rho) + \frac{c+d}{2\alpha-\beta} \rho. \quad (4.23)$$

And of course, with the charge parameter $q = 0$, there is no electric field. Hence, we have fully solved Case 1.

Case 2: $\Lambda = 0$

Here we have a system with a massless scalar field and an EM field coupled to a Liouville potential. The solutions are:

$$\begin{aligned} \text{Metric:} \quad ds^2 = & -e^{-\frac{2\alpha(d-c)}{2\alpha-\beta}\rho} H(\rho)^{\frac{D-3}{(D-3+(D-2)\alpha^2)}} dt^2 + e^{-2d\rho} H(\rho)^{-\frac{1}{(D-3+(D-2)\alpha^2)}} d\rho^2 \\ & + \sum_{i=1}^n e^{-\frac{2}{D-2}\left(\frac{\alpha(d+c)-\beta d}{2\alpha-\beta}\right)\rho} H(\rho)^{-\frac{1}{(D-3+(D-2)\alpha^2)}} dx_i^2, \end{aligned} \quad (4.24)$$

$$\begin{aligned} \text{Constraint:} \quad & \frac{b^2(D-2)(D-2)(\beta-2\alpha)^2}{\alpha^2(D-2)+D-3} = c^2(\alpha^2(3D-7)-D+2) \\ & + 2cd(-3\alpha^2+5\alpha\beta+\alpha^2D-2\alpha\beta D+D-2) \\ & + d^2(\alpha^2+2\alpha\beta-(\alpha^2+1)D+\beta^2(D-3)+2), \end{aligned} \quad (4.25)$$

$$\text{Scalar Field:} \quad \varphi = -\frac{\alpha}{2\left(\frac{D-3}{D-2}+\alpha^2\right)} \ln H(\rho) + \frac{(d-c)\rho}{2\alpha-\beta}, \quad (4.26)$$

$$\text{Electric Field:} \quad E = -\frac{b^2(D-2)\text{csch}^2(b\rho)}{2q(D-3+\alpha^2(D-2))}, \quad (4.27)$$

where $H(\rho) = \frac{2q^2}{b^2} \left(\frac{D-3}{D-2} + \alpha^2\right) \sinh^2(b\rho)$.

Case 3: $\alpha = \beta = 1, D = 3$

In this case, we are in (2+1)-dimensions with a massless scalar field, EM field and cosmological constant with specific values for the coupling constants. This case corresponds to the (2+1)-dimension low-energy action obtained from string theory [20]. The solutions accordingly are:

$$\begin{aligned} \text{Metric:} \quad ds^2 = & -e^{-2b_3\rho} \frac{1}{M(\rho)} dt^2 + e^{-2b_3\rho} \left(\frac{2q^2 \sinh^2(b_2\rho)}{b_2^2 M(\rho)}\right) d\rho^2 \\ & + \left(\frac{2q^2 \sinh^2(b_2\rho)}{b_2^2 M(\rho)}\right) dx^2, \end{aligned} \quad (4.28)$$

$$\text{Constraint:} \quad b_1^2 - b_2^2 - b_3^2 = 0, \quad (4.29)$$

$$\text{Scalar Field:} \quad \varphi = b_3 \rho + \ln \left(\frac{b_2 \sqrt{M(\rho)}}{\sqrt{2}q \sinh(b_2 \rho)} \right), \quad (4.30)$$

$$\text{Electric Field:} \quad E = -\frac{b_2^2 \text{csch}^2(b_2 \rho)}{2q}, \quad (4.31)$$

with

$$M(\rho) = \begin{cases} \frac{2\Lambda}{b_1^2} \cosh^2(b_1 \rho), & \text{for } \Lambda > 0, \\ \frac{2\Lambda}{b_1^2} \sinh^2(b_1 \rho), & \text{for } \Lambda < 0. \end{cases} \quad (4.32)$$

Case 4: $\alpha\beta = 1$, $D = 3$

Again, we are in (2+1)-dimensions with a massless scalar field, EM field and cosmological constant but now the coupling constants are allowed to take on any positive values which satisfies $\alpha\beta = 1$. The solutions are:

$$\text{Metric:} \quad ds^2 = -e^{-\frac{2\alpha^2(c\rho)}{2\alpha^2-1}} J(\rho)^{\frac{-\alpha^2}{2\alpha^2-1}} dt^2 \quad (4.33)$$

$$\begin{aligned} &+ e^{\frac{2(c\rho)}{2\alpha^2-1}} \left(\frac{2q^2\alpha^2}{b_2^2} \sinh^2(b_2 \rho) \right)^{\frac{1}{\alpha^2}} J(\rho)^{-\frac{2\alpha^2}{2\alpha^2-1}} d\rho^2 \\ &+ e^{\frac{2(\alpha^2-1)(c\rho)}{2\alpha^2-1}} \left(\frac{2q^2\alpha^2}{b_2^2} \sinh^2(b_2 \rho) \right)^{\frac{1}{\alpha^2}} J(\rho)^{-\frac{\alpha^2}{2\alpha^2-1}} dx^2, \end{aligned} \quad (4.34)$$

$$\text{Constraint:} \quad \alpha^4 b_1^2 + (1 - 2\alpha^2) b_2^2 - \alpha^4 c^2 = 0, \quad (4.35)$$

$$\text{Scalar Field:} \quad \varphi = \frac{\alpha}{2(2\alpha^2 - 1)} \ln J(\rho) - \frac{1}{\alpha} \ln \left(\frac{q\alpha\sqrt{2}}{b_2} \sinh(b_2 \rho) \right) + \frac{\alpha c \rho}{2\alpha^2 - 1}, \quad (4.36)$$

$$\text{Electric Field:} \quad E = -\frac{b_2^2 \text{csch}^2(b_2 \rho)}{2\alpha^2 q}, \quad (4.37)$$

Where

$$J(\rho) = \begin{cases} \frac{2\Lambda(2\alpha^2-1)}{\alpha^2 b_1^2} \cosh^2(b_1 \rho), & \text{for } 2\Lambda(2\alpha^2 - 1) > 0, \\ \frac{2\Lambda(1-2\alpha^2)}{\alpha^2 b_1^2} \sinh^2(b_1 \rho), & \text{for } 2\Lambda(2\alpha^2 - 1) < 0. \end{cases} \quad (4.38)$$

Note that Case 3 is a subset of Case 4, and we recover all the fields identically when setting $\alpha = 1$ above.

One might also consider setting $\alpha = 0, \beta \rightarrow \infty$ to recover Case 2 with $\alpha = 0$ as well. Unfortunately we have α factors in the denominator in the above solution and therefore this cannot easily reduce back to Case 2.

Chapter 5

Singularities

Up to this point, we have basically only solved a long mathematical problem and have yet to extract any real meaning or interpretation from the solutions we have found. In this chapter, we will take a look at the singularities presented in the solutions for each case and check to see whether these are merely coordinate artefacts or true singularities. We shall mention briefly that the Cosmic Censorship Hypothesis conjectured by Roger Penrose [21][22] postulates that the existence of a naked singularity (one that is not hidden within event horizons) is physically unrealistic. Therefore while there are many known solutions which violate this hypothesis, such as the AdS naked singularity or the disappearing event horizons in the Kerr metric [23] and the Reissner-Nordström metric [24], they are often derived from overly-simplified assumptions or unphysical systems which cannot exist in our universe. We will see in this chapter that some of our solutions will violate the Cosmic Censorship Hypothesis, and some of them will not. Those that do not may then be called black hole solutions.

5.1 Making Sense of the ρ -Coordinate

Given that we began with a general metric ansatz dependent only on one coordinate with no assumptions to the topology or geometry of our spacetime, all of our coordinates have no intrinsic meaning attached to them. This means that we have no idea of where the “origin” lies in our spacetime, and which direction of the ρ -coordinate points to being near or far from the origin.

Therefore we need to, first, have a reference spacetime to “align” ourselves to known

coordinate systems so that we may then begin to understand the geometry of all our solutions. One way to do this is to compare our solutions for physical quantities like the electric or scalar fields with others' (provided that the systems in consideration are the same) to obtain a coordinate transformation. Fortunately, Lim's solution back in Chapter 3 is derived from a similar system as Case 1: $q = 0$, with the exception of the Liouville potential coupled to the cosmological constant.

It is then a simple matter of setting $\beta = 0$ to recover the exact same system. Equating our obtained scalar field in Eq. (4.23) with Lim's in Eq. (3.15), we get:

$$\begin{aligned} \text{Our scalar field} \Big|_{\beta=0} &= \text{Lim's scalar field} \\ \underbrace{\frac{c+d}{\alpha}}_C \rho &= \underbrace{\frac{1}{2} \sqrt{\frac{n(1-v^2)}{n+1}}}_B \ln \left(1 - \left(\frac{r_0}{r} \right)^{n+1} \right) \\ \rho &= \frac{B}{C} \ln \left(1 - \left(\frac{r_0}{r} \right)^{n+1} \right). \end{aligned} \quad (5.1)$$

In Lim's coordinate, $r = 0$ is the origin and $r \rightarrow \infty$ is far away from the origin. With this mapping, we have $\rho = 0 \mapsto r = \infty$ and $\rho = -\infty \mapsto r = r_0$ (the " $= \infty$ " is simply an informal equation). As one might notice, our ρ -coordinate is unable to probe the region $0 < r < r_0$ which is where Lim's coordinate also sort of breaks down due to his expression for the scalar field. Nonetheless, it is still valid to point out singularities within the region inside the event horizon, r_0 , if they appear in the metric or curvature scalars. This is very much akin to the swapping of the time and radial coordinates in the Schwarzschild solution

$$ds_{\text{Schwarzschild}}^2 = - \left(1 - \frac{2m}{r} \right) dt^2 + \left(1 - \frac{2m}{r} \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (5.2)$$

when entering the region $r < 2m$ as seen above. The coordinate system also breaks down but we may still deduce points of divergence.

Therefore it makes sense to search for singularities in our solution in the r -coordinates instead. As all our solutions contain hyperbolic functions, we simply need to express the hyperbolic functions in the r -coordinates. We first set $b = \frac{C}{B}$ which is the prefactor in all our solutions, then Eq. (5.1) may be rearranged to obtain

$$e^{b\rho} = \frac{r^{n+1} - r_0^{n+1}}{r^{n+1}}, \quad (5.3)$$

$$e^{-b\rho} = \frac{r^{n+1}}{r^{n+1} - r_0^{n+1}}. \quad (5.4)$$

The hyperbolic functions may then be expressed as

$$\sinh(b\rho) = \frac{1}{2} \left(\frac{r_0^{n+1} (r_0^{n+1} - 2r^{n+1})}{r^{n+1} (r^{n+1} - r_0^{n+1})} \right), \quad (5.5a)$$

$$\operatorname{csch}(b\rho) = 2 \left(\frac{r^{n+1} (r^{n+1} - r_0^{n+1})}{r_0^{n+1} (r_0^{n+1} - 2r^{n+1})} \right), \quad (5.5b)$$

$$\cosh(b\rho) = \frac{1}{2} \left(\frac{r_0^{n+1} (r_0^{n+1} - 2r^{n+1}) + 2r^{2n+2}}{r^{n+1} (r^{n+1} - r_0^{n+1})} \right), \quad (5.5c)$$

$$\operatorname{sech}(b\rho) = 2 \left(\frac{r^{n+1} (r^{n+1} - r_0^{n+1})}{r_0^{n+1} (r_0^{n+1} - 2r^{n+1}) + 2r^{2n+2}} \right), \quad (5.5d)$$

$$\tanh(b\rho) = \frac{r_0^{n+1} (r_0^{n+1} - 2r^{n+1})}{r_0^{n+1} (r_0^{n+1} - 2r^{n+1}) + 2r^{2n+2}}, \quad (5.5e)$$

$$\operatorname{coth}(b\rho) = \frac{r_0^{n+1} (r_0^{n+1} - 2r^{n+1}) + 2r^{2n+2}}{r_0^{n+1} (r_0^{n+1} - 2r^{n+1})}. \quad (5.5f)$$

Thus we can see that

- \sinh and \cosh diverges at $r = 0$ and $r = r_0$,
- csch and coth diverges at $r = 2^{-\frac{1}{n+1}} r_0 (< r_0)$,
- sech and \tanh do not diverge.

Note that this is merely a qualitative approach and is vastly different from how Maki analyzed her solutions. Since her solutions are all in (2+1)-dimensions, she simply assumed rotational symmetry by associating $g_{xx} dx^2 \mapsto r^2 d\phi^2$, where r and ϕ are the normal radial and azimuthal coordinates. However, because we are now dealing with arbitrary dimensions, we do not have the luxury of this mapping. The additional dimensions, as mentioned before, serve as a isotropic, flat background and we may not simply associate this with D -dimensional spherical symmetry. With this, we will proceed to investigate the coordinate and curvature singularities in all our solutions in the following section.

5.2 Coordinate and Curvature Singularities

Coordinate singularities refer to diverging terms in the metric components, whereas curvature singularities may be deduced from a diverging Ricci or Kretschmann scalar [25].

In Eq. (2.15), we have found the Ricci scalar to be

$$R = e^{-2H} \sum_{i=0}^n \left(2H'F'_i - 2F''_i - F_i'^2 - F'_i \sum_{j=0}^n F'_j \right).$$

Including the new assumptions made in the previous chapter, we first rewrite the formula to be

$$\begin{aligned} R &= e^{-2H} \sum_{i=0}^n \left(2F'_i \sum_{k=0}^n F'_k - 2F''_i - F_i'^2 - F'_i \sum_{j=0}^n F'_j \right) \\ &= e^{-2H} \sum_{i=0}^n (F'_i F'_0 + (D-2)F'_i F'_{i \neq 0} - 2F''_i - F_i'^2) \\ &= e^{-2H} \left[\underbrace{(D-2)F'_0 F'_{i \neq 0} - 2F''_0}_{i=0 \text{ term}} + (D-2)F'_{i \neq 0} F'_0 + (D-2)^2 F_{i \neq 0}'^2 - 2(D-2)F''_{i \neq 0} \right. \\ &\quad \left. - (D-2)F_{i \neq 0}'^2 \right] \\ &= e^{-2H} \left[2(D-2)F'_0 F'_{i \neq 0} - 2F''_0 + (D^2 - 5D + 6)F_{i \neq 0}'^2 - 2(D-2)F''_{i \neq 0} \right]. \end{aligned} \quad (5.6)$$

We will use Mathematica to compute the Ricci scalars and only show the final results in the sections below.

5.2.1 Case 1a: $q = 0$, $2\Lambda \left(\beta^2 - \frac{D-1}{D-2} \right) < 0$

The final metric was found to be

$$\begin{aligned} ds^2 &= - e^{2\left(\frac{\alpha(c-d)-\beta c}{2\alpha-\beta}\right)\rho} G(\rho)^{\frac{1}{D-2} \left(\frac{1}{\beta^2 - \frac{D-1}{D-2}} \right)} dt^2 + e^{2(c-d)\rho} G(\rho)^{\frac{D-1}{D-2} \left(\frac{1}{\beta^2 - \frac{D-1}{D-2}} \right)} d\rho^2 \\ &\quad + \sum_{i=1}^n e^{-\frac{2}{D-2} \left(\frac{\alpha(d-c)-\beta d}{2\alpha-\beta} \right)\rho} G(\rho)^{\frac{1}{D-2} \left(\frac{1}{\beta^2 - \frac{D-1}{D-2}} \right)} dx_i^2, \end{aligned}$$

with $G(\rho) = \frac{2\Lambda}{b^2} \left(\frac{D-1}{D-2} - \beta^2 \right) \cosh^2(b\rho)$. The Ricci scalar is computed to be

$$\begin{aligned}
 R = & 2^{\frac{1-D}{\beta^2(D-2)-D+1}} e^{2\rho(d-c)} \left(\operatorname{sech}^2(b\rho) \frac{b^2(D-2)}{\Lambda(\beta^2(-(D-2)) + D-1)} \right)^{\frac{D-1}{\beta^2(D-2)-D+1}} \\
 & \times \left[\frac{2b(D-2)(c-d) \tanh(b\rho)}{\beta^2(D-2) - D + 1} + \frac{b^2(D-1) ((D-2)\beta^2(D-2)) \operatorname{sech}^2(b\rho) + D-2}{(\beta^2(D-2) - D + 1)^2} \right. \\
 & \left. + \frac{(\alpha c + d(\beta - \alpha))(\alpha(3D-7)(c-d) + \beta(d(D-3) - 2c(D-2)))}{(D-2)(\beta - 2\alpha)^2} \right]. \quad (5.7)
 \end{aligned}$$

In this case, the cosh function in the metric indicates coordinate singularities at $r = 0$ and $r = r_0$ while the Ricci scalar indicates a curvature singularity only at $r = 0$ from the e^ρ prefactor. The sech and tanh functions do not contribute diverging terms. Therefore we have a true singularity at $r = 0$ and an event horizon at $r = r_0$, hence this is a black hole solution.

5.2.2 Case 1b: $q = 0$, $2\Lambda \left(\beta^2 - \frac{D-1}{D-2} \right) > 0$

Here, we have instead $G(\rho) = \frac{2\Lambda}{b^2} \left(\beta^2 - \frac{D-1}{D-2} \right) \sinh^2(b\rho)$. The Ricci scalar is computed to be

$$\begin{aligned}
 R = & 2^{\frac{1-D}{\beta^2(D-2)-D+1}} e^{2\rho(d-c)} \left(\operatorname{csch}^2(b\rho) \frac{b^2(D-2)}{\Lambda(\beta^2(D-2) - D + 1)} \right)^{\frac{D-1}{\beta^2(D-2)-D+1}} \\
 & \times \left[\frac{(\alpha c + d(\beta - \alpha))(\alpha(3D-7)(c-d) + \beta(d(D-3) - 2c(D-2)))}{(D-2)(\beta - 2\alpha)^2} \right. \\
 & + \frac{b^2(D-1) \operatorname{csch}^2(b\rho) ((D-2) \cosh(2b\rho) + 4\beta^2(D-2) - 3D + 2)}{2(\beta^2(D-2) - D + 1)^2} \\
 & \left. + \frac{2b(D-2)(c-d) \coth(b\rho)}{\beta^2(D-2) - D + 1} \right]. \quad (5.8)
 \end{aligned}$$

The sinh function in the metric indicates coordinate singularities at $r = 0$ and $r = r_0$ while the Ricci scalar indicates two curvature singularity at $r = 0$ from the e^ρ and $r = 2^{-\frac{1}{n+1}} r_0$ from the csch and coth functions. The cosh function will be cancelled off if we open up the brackets

$$\begin{aligned}
 \operatorname{csch}^2(b\rho) \cosh(2b\rho) &= \operatorname{csch}(b\rho)^2 (2 \cosh^2(b\rho) - 1) \\
 &= \coth^2(b\rho) - \operatorname{csch}^2(b\rho)
 \end{aligned}$$

$$= 1, \quad (5.9)$$

and hence does not contribute any diverging term. Therefore, we have two singularities covered by the event horizon and thus, we again have a black hole solution.

5.2.3 Case 2: $\Lambda = 0$

For $\Lambda = 0$, the final metric solution is

$$ds^2 = - e^{-\frac{2\alpha(d-c)}{2\alpha-\beta}\rho} H(\rho)^{\frac{D-3}{(D-3+(D-2)\alpha^2)}} dt^2 + e^{-2d\rho} H(\rho)^{-\frac{1}{(D-3+(D-2)\alpha^2)}} d\rho^2 \\ + \sum_{i=1}^n e^{-\frac{2}{D-2}\left(\frac{\alpha(d+c)-\beta d}{2\alpha-\beta}\right)\rho} H(\rho)^{-\frac{1}{(D-3+(D-2)\alpha^2)}} dx_i^2,$$

with $H(\rho) = \frac{2q^2}{b^2} \left(\frac{D-3}{D-2} + \alpha^2 \right) \sinh^2(b\rho)$. The Ricci scalar is computed to be

$$R = e^{\frac{4\alpha\rho(d-c)}{2\alpha-\beta}} \left(\operatorname{csch}^2(b\rho) \frac{b^2}{2q^2 \left(\alpha^2 + \frac{D-3}{D-2} \right)} \right)^{\left(\frac{1}{\alpha^2(D-2)+D-3} \right)} \\ \times \left[\frac{2ab(D-2)(c-d) \coth(b\rho)}{(2\alpha-\beta)(\alpha^2(D-2)+D-3)} + \frac{\alpha^2(3D-7)(c-d)^2}{(D-2)(\beta-2\alpha)^2} \right. \\ \left. - \frac{b^2 \operatorname{csch}^2(b\rho) \left((D-3)(D-2) \cosh(2b\rho) + D^2 - 4\alpha^2(D-2) - 9D + 18 \right)}{2(\alpha^2(D-2)+D-3)^2} \right]. \quad (5.10)$$

The sinh function in the metric indicates coordinate singularities at $r = 0$ and $r = r_0$ while the Ricci scalar indicates two curvature singularity at $r = 0$ from the e^ρ and $r = 2^{-\frac{1}{n+1}} r_0$ from the csch and coth functions. Again, the cosh function does not contribute to any diverging term after multiplying out the csch^2 factor. Accordingly, we have two singularities covered by the event horizon which is also a black hole solution.

5.2.4 Case 3: $\alpha = \beta = 1, D = 3$

The metric solution is

$$ds^2 = - e^{-2b_3\rho} \left(\frac{b_1^2}{2\Lambda} \operatorname{sech}^2(b_1\rho) \right) dt^2 + e^{-2b_3\rho} \left(\frac{q^2 b_1^4 \sinh^2(b_2\rho)}{2b_2^2 \Lambda^2 \cosh^4(b_1\rho)} \right) d\rho^2$$

$$+ \left(\frac{q^2 b_1^2 \sinh^2(b_2 \rho)}{b_2^2 \Lambda \cosh^2(b_1 \rho)} \right) dx^2,$$

and the Ricci scalar is computed to be

$$\begin{aligned} R = & \frac{2b_2^2}{b_1^4 q^2} \Lambda^2 e^{2(b_3 \rho + b_4)} \cosh^2(b_1 \rho) \operatorname{csch}^2(b_2 \rho) \\ & \times \left(b_1^2 (\cosh(2b_1 \rho) + 3) + b_1 \sinh(2b_1 \rho) (b_3 - b_2 \coth(b_2 \rho)) \right. \\ & \left. + 2b_2 \cosh^2(b_1 \rho) (b_2 \operatorname{csch}^2(b_2 \rho) - b_3 \coth(b_2 \rho)) \right). \end{aligned} \quad (5.11)$$

In this case, the description of the spacetime is rather messy because we have the different b_1 , b_2 and b_3 in the hyperbolic arguments. This produces a multitude of singularities which becomes meaningless. Instead, we can examine a special case of this spacetime where $b_3 = 0$. The constraint equation from Eq. (4.29) which was

$$b_1^2 - b_2^2 - b_3^2 = 0,$$

then reduces to $b_1 = b_2 \equiv b$ and the metric simplifies greatly to

$$\begin{aligned} ds_{\text{simplified}}^2 = & - \frac{b^2}{2\Lambda} \operatorname{sech}^2(b\rho) dt^2 + \frac{q^2 b^2}{2\Lambda^2} \operatorname{sech}^2(b\rho) \tanh^2(b\rho) d\rho^2 \\ & + \frac{q^2}{\Lambda} \tanh^2(b\rho) dx^2. \end{aligned} \quad (5.12)$$

Correspondingly, the Ricci scalar becomes

$$R = \frac{4\Lambda^2}{q^2} (\coth^4(b\rho) + \coth^2(b\rho)). \quad (5.13)$$

Thus, we gather that there are no coordinate singularities in the metric and there exists one naked singularity at $r = 2^{-\frac{1}{n+1}} r_0$ for this special choice of parameters.

5.2.5 Case 4a: $\alpha\beta = 1$, $D = 3$, $2\alpha^2 > 1$

The metric solution is

$$ds^2 = - e^{-\frac{2\alpha^2(c_2\rho)}{2\alpha^2-1}} J(\rho)^{\frac{-\alpha^2}{2\alpha^2-1}} dt^2 + e^{\frac{2(c_2\rho)}{2\alpha^2-1}} \left(\frac{2q^2\alpha^2}{b_2^2} \sinh^2(b_2\rho) \right)^{\frac{1}{\alpha^2}} J(\rho)^{-\frac{2\alpha^2}{2\alpha^2-1}} d\rho^2$$

$$+ e^{\frac{2(\alpha^2-1)(c_2\rho)}{2\alpha^2-1}} \left(\frac{2q^2\alpha^2}{b_2^2} \sinh^2(b_2\rho) \right)^{\frac{1}{\alpha^2}} J(\rho)^{-\frac{\alpha^2}{2\alpha^2-1}} dx^2, \quad (5.14)$$

where $J(\rho) = \frac{2\Lambda(2\alpha^2-1)}{\alpha^2 b_1^2} \cosh^2(b_1\rho)$. The Ricci scalar is computed to be

$$\begin{aligned} R = & \frac{2^{\frac{2\alpha^2}{2\alpha^2-1}-\frac{1}{\alpha^2}}}{(\alpha-2\alpha^3)^2} e^{\frac{2c_2\rho}{2\alpha^2-1}} \left(\frac{\alpha q \sinh(b_2\rho)}{b_2} \right)^{-\frac{2}{\alpha^2}} \left(\frac{(2-\frac{1}{\alpha^2})\Lambda \cosh^2(b_1\rho)}{b_1^2} \right)^{\frac{2\alpha^2}{2\alpha^2-1}} \\ & \times \left[2\alpha^2 b_1 \tanh(b_1\rho) ((1-2\alpha^2) b_2 \coth(b_2\rho) + \alpha^2 c_2) \right. \\ & + 2(2\alpha^2-1) b_2 ((2\alpha^2-1) b_2 \operatorname{csch}^2(b_2\rho) - \alpha^2 c_2 \coth(b_2\rho)) \\ & \left. + \alpha^4 b_1^2 \operatorname{sech}^2(b_1\rho) (7\alpha^2 + \alpha^2 \cosh(2b_1\rho) - 4) - 2(\alpha^2-1) \alpha^4 c_2^2 \right]. \quad (5.15) \end{aligned}$$

Again, it would be sensible to simplify this using the constraint in Eq. (4.35) which was

$$\alpha^4 b_1^2 + (1-2\alpha^2) b_2^2 - \alpha^4 c^2 = 0.$$

The choice of $c = 0$ and $b_1 = b_2 \equiv b$ leaves the constraint equation to become

$$\begin{aligned} \frac{\alpha^2}{2\alpha^2-1} &= \frac{1}{\alpha^2} \\ \Rightarrow \alpha^2 &= 1. \quad (5.16) \end{aligned}$$

Surprisingly, it turns out that this convenient choice of parameters also results in the special case where $\alpha = 1$ which should return us back to Case 3. The metric becomes

$$\begin{aligned} ds_{\text{simplified}}^2 = & - \left(\frac{b^2}{2\Lambda} \operatorname{sech}^2(b\rho) \right) dt^2 + \left(\frac{q^2 b^2}{2\Lambda^2} \operatorname{sech}^2(b\rho) \tanh^2(b\rho) \right) d\rho^2 \\ & + \left(\frac{q^2}{\Lambda} \tanh^2(b\rho) \right) dx^2, \quad (5.17) \end{aligned}$$

which is precisely that in Eq. (5.11), and the Ricci scalar simplifies to

$$\begin{aligned} R = & 2 \left(\frac{q \sinh(b\rho)}{b} \right)^{-2} \left(\frac{\Lambda \cosh^2(b\rho)}{b^2} \right)^2 \left[-2b^2 \tanh(b\rho) \coth(b\rho) + 2b^2 \operatorname{csch}^2(b\rho) \right. \\ & \left. + b^2 \operatorname{sech}^2(b\rho) (7 + \cosh(2b\rho) - 4) \right] \end{aligned}$$

$$\begin{aligned}
 &= 2 \frac{\operatorname{csch}^2(b\rho)}{q^2} \Lambda^2 \cosh^4(b\rho) \left[-2 + 2\operatorname{csch}^2(b\rho) + \operatorname{sech}^2(b\rho) (3 + 2\cosh^2(b\rho) - 1) \right] \\
 &= \frac{4\Lambda^2}{q^2} \coth^2(b\rho) \cosh^2(b\rho) \left[\operatorname{csch}^2(b\rho) + \operatorname{sech}^2(b\rho) \right] \\
 &= \frac{4\Lambda^2}{q^2} (\coth^4(b\rho) + \coth^2(b\rho)), \tag{5.18}
 \end{aligned}$$

and true enough, we obtain exactly Eq. (5.13) which shows that our solutions are coherent and concise.

5.2.6 Case 4b: $\alpha\beta = 1$, $D = 3$, $2\alpha^2 < 1$

Here, we have instead $J(\rho) = \frac{2\Lambda(1-2\alpha^2)}{\alpha^2 b_1^2} \sinh^2(b_1\rho)$. The Ricci scalar is computed to be

$$\begin{aligned}
 R &= \frac{2^{\frac{2\alpha^2}{2\alpha^2-1} - \frac{1}{\alpha^2}}}{(\alpha - 2\alpha^3)^2} e^{\frac{2c_2\rho}{2\alpha^2-1}} \left(\frac{\alpha q \sinh(b_2\rho)}{b_2} \right)^{-\frac{2}{\alpha^2}} \left(\frac{(\frac{1}{\alpha^2} - 2) \Lambda \sinh^2(b_1\rho)}{b_1^2} \right)^{\frac{2\alpha^2}{2\alpha^2-1}} \\
 &\times \left[\alpha^4 (2b_1 c_2 \coth(b_1\rho) + b_1^2 \operatorname{csch}^2(b_1\rho) (-7\alpha^2 + \alpha^2 \cosh(2b_1\rho) + 4) - 2(\alpha^2 - 1) c_2^2) \right. \\
 &\quad \left. - 2(2\alpha^2 - 1) \alpha^2 b_2 \coth(b_2\rho) (b_1 \coth(b_1\rho) + c_2) + 2(1 - 2\alpha^2)^2 b_2^2 \operatorname{csch}^2(b_2\rho) \right]. \tag{5.19}
 \end{aligned}$$

Unfortunately, since this case shares the exact same constraint equation as Case 4a, we cannot make the same convenient choices for the parameters because $\alpha = 1$ is not valid for $2\alpha^2 < 1$ to be true. We may first keep $b_1 = b_2 = b$, and try instead to set $c^2 = 2b^2$ which simplifies the constraint equation to

$$\begin{aligned}
 \frac{\alpha^2}{(1 - 2\alpha^2)} &= \frac{1}{\alpha^2} \\
 \Rightarrow \alpha^2 &= -1 \pm \sqrt{2}. \tag{5.20}
 \end{aligned}$$

But this is also not a valid solution because α , by definition, must be real. Therefore we do not have any clear choice for the parameters which could greatly simplify our solutions as in the previous cases. Thus, we shall just leave this case as it is and conclude this section on singularities.

Chapter 6

Limiting Behavior and Geodesics of $q = 0$ and $\Lambda = 0$ Solutions

We have previously shown that the solutions in Cases 1 and 2 are black hole solutions. This makes them physically interesting and we therefore shift our focus to these solutions and investigate their limiting behavior and geodesics

6.1 Limiting Cases

6.1.1 Case 1: $q = 0$, $\beta = 0$ to AdS Spacetime

Previously, we have found that a singularity at $r = 0$ is hidden within the event horizon at $r = r_0$ for $2\Lambda \left(\beta^2 - \frac{D-1}{D-2}\right) < 0$. However, we deduced this qualitatively without being meticulous with the mess of prefactors in Eq. (5.7). It is possible that certain values of the parameters could produce cancellations and change the hyperbolic functions in the Ricci scalar according to the hyperbolic function identities. Here, we will consider the special case where $\beta = 0$, which is the same as removing the Liouville coupling from the cosmological constant. The system then becomes identical to Lim's again and we can now check the behavior of the singularities in this special case. Taking a look at the scalar field from Eq. (4.23),

$$\begin{aligned}\varphi &= -\frac{\beta}{2\left(\beta^2 - \frac{D-1}{D-2}\right)} \ln G_{\pm}(\rho) + \frac{c+d}{2\alpha - \beta} \rho, \\ \Rightarrow \varphi_{\beta=0} &= \frac{c+d}{2\alpha} \rho,\end{aligned}\tag{6.1}$$

we may further split this into two cases:

1. Setting $c = d$ will help us eliminate one arbitrary constant and could perhaps recover Lim's AdS naked planar singularity. This choice is also motivated to help cancel some $(c - d)$ factors seen in the Ricci scalar in Eq. (5.7).
2. Setting $c = -d$ will turn off the scalar field and we should recover back a metric which reflects a similar AdS spacetime as Ren's.

6.1.1.1 For $c = -d$, Zero Scalar Field

The metric from Eq. (4.21) simplifies to

$$\begin{aligned}
 ds^2 = & - e^{2c\rho} \left(\frac{2\Lambda}{b^2} \left(\frac{D-1}{D-2} \right) \cosh^2(b\rho) \right)^{-\frac{1}{D-1}} dt^2 + e^{4c\rho} \left(\frac{2\Lambda}{b^2} \left(\frac{D-1}{D-2} \right) \cosh^2(b\rho) \right)^{-1} d\rho^2 \\
 & + \sum_{i=1}^n e^{\frac{2c\rho}{D-2}} \left(\frac{2\Lambda}{b^2} \left(\frac{D-1}{D-2} \right) \cosh^2(b\rho) \right)^{-\frac{1}{D-1}} dx_i^2.
 \end{aligned} \tag{6.2}$$

The constraint from Eq. (4.20) also simplifies to

$$c^2 = \frac{b^2(D-2)^2}{(D-1)^2}. \tag{6.3}$$

Plugging the parameters into the Ricci scalar gives us

$$\begin{aligned}
 R &= \frac{\Lambda}{b^2(D-2)^2} \left[b^2(D-2) \left((D-2) \cosh(2b\rho) + 3D-2 \right) - 2c^2(D-1)^2 \cosh^2(b\rho) \right] \\
 &= \Lambda \left[\frac{(D-2)(2 \cosh^2(b_1\rho) - 1) + 3D-2}{D-2} - 2 \cosh^2(b_1\rho) \right] \\
 &= \frac{2D\Lambda}{D-2},
 \end{aligned} \tag{6.4}$$

where we have used the constraint equation in the first to second step. Indeed, we see that removing the Liouville coupling and turning off the scalar field for our solution recovers the exact Ricci scalar for flat AdS spacetime. This shows that our solution for Case 1 is a valid spacetime since it reproduces the expected result in this particular limit.

6.1.1.2 For $c = d$, Non-Zero Scalar Field

Here, the metric simplifies to

$$ds^2 = - \left(\frac{2\Lambda}{b^2} \left(\frac{D-1}{D-2} \right) \cosh^2(b\rho) \right)^{-\frac{1}{D-1}} dt^2 + \left(\frac{2\Lambda}{b^2} \left(\frac{D-1}{D-2} \right) \cosh^2(b\rho) \right)^{-1} d\rho^2 + \sum_{i=1}^n \left(\frac{2\Lambda}{b^2} \left(\frac{D-1}{D-2} \right) \cosh^2(b\rho) \right)^{-\frac{1}{D-1}} dx_i^2, \quad (6.5)$$

and the constraint is instead

$$c_2^2 = \frac{\alpha^2 b^2 (D-2)}{D-1}. \quad (6.6)$$

However, since the parameters α and c have become cancelled off in the metric, the constraint equation basically implies that b can take on any real value (not imaginary because c^2 and α^2 are necessarily positive). The Ricci scalar from Eq. (5.7) becomes

$$R = 2 \left(\operatorname{sech}^2(b\rho) \frac{b^2(D-2)}{\Lambda(D-1)} \right)^{-1} \left(\frac{b^2(D-1)(D \operatorname{sech}^2(b\rho) + D-2)}{(D-1)^2} \right) = \underbrace{\frac{2\Lambda D}{(D-2)}}_{\text{AdS part}} + \underbrace{2\Lambda (\cosh^2(b\rho))}_{\text{presence of scalar field}}. \quad (6.7)$$

The additional term created by the non-zero scalar field is similar to the situation occurring back in Chapter 3, where we calculated the $R_{\rho\rho}$ Ricci tensor component for Lim's metric in Eq. (3.19) and found it to be the AdS part with a scalar field contribution. Again, using the coordinate mapping, the cosh function tells us that we have curvature singularities at $r = 0$ and $r = r_0$. This is the same result obtained by Lim, where the presence of a scalar field causes the original event horizon of the AdS black hole at $r = r_0$ to turn into a singularity.

6.1.1.3 A Final Note

We have made several attempts to perform coordinate transformations on our $q = 0$ metric solution to see if it maps precisely back to the coordinates used by Lim and Ren. The association of the scalar fields performed in the previous chapter fails to reproduce Lim's metric from ours. One possible reason might be due to the incomplete

mapping of domains between the r -coordinate and ρ -coordinate, since we are unable to probe the region $0 < r < r_0$ as mentioned before. Therefore we cannot say for sure as to whether we might have found the same solution in a different coordinate system, or a whole new solution altogether.

Nevertheless, we may state with confidence that we have successfully introduced a further generalization of Lim's solution in Einstein-Dilaton gravity by introducing the β coupling parameter and showed that it still reproduces the right singularities in the case where $\beta = 0$.

6.1.2 Case 2: $\Lambda = 0$ to the Melvin Solution

There are a wide variety of charged solutions in EMD gravity – some of the simpler ones include the Bañados, Teitelboim and Zanelli (BTZ) black hole [26] and the aforementioned Reissner-Nordström metric. Unfortunately, the BTZ black hole is a solution for the case of a negative cosmological constant which does not apply for Case 2. On the other hand, the Reissner-Nordström metric possesses spherical symmetry which is absent in our solution. Taking a look at our previously found metric from Eq. (4.24),

$$\begin{aligned}
 ds^2 &= - e^{-\frac{2\alpha(d-c)}{2\alpha-\beta}\rho} H(\rho)^{\frac{D-3}{(D-3+(D-2)\alpha^2)}} dt^2 + e^{-2d\rho} H(\rho)^{-\frac{1}{(D-3+(D-2)\alpha^2)}} d\rho^2 \\
 &\quad + \sum_{i=1}^n e^{-\frac{2}{D-2}\left(\frac{\alpha(d+c)-\beta d}{2\alpha-\beta}\right)\rho} H(\rho)^{-\frac{1}{(D-3+(D-2)\alpha^2)}} dx_i^2, \\
 H(\rho) &= \frac{2q^2}{b^2} \left(\frac{D-3}{D-2} + \alpha^2 \right) \sinh^2(b\rho),
 \end{aligned}$$

we notice that the exponent of the $H(\rho)$ function in the $g_{\rho\rho}$ and g_{ii} terms are identical. The Melvin solution [10] coincidentally exhibits a very similar appearance. In addition, it solves the exact same EMD action in which we are considering for Case 2 – where $\Lambda = 0$. In 4-dimensions and using cylindrical coordinates, the electrically-charged Melvin solution (in our symbol convention) is given by [8]

$$ds_{\text{Melvin}}^2 = L(r)^{\frac{2}{1+2\alpha^2}} (dr^2 + dz^2 - dt^2) + L(r)^{-\frac{2}{1+2\alpha^2}} r^2 d\phi^2, \quad (6.8)$$

where

$$e^{-2\alpha(\varphi-\varphi_0)} = L(r)^{\frac{4\alpha^2}{1+2\alpha^2}}, \quad A = e^{\alpha\varphi_0} \frac{Qr^2}{2L(r)},$$

$$L(r) = 1 + \left(\frac{1 + 2\alpha^2}{4} \right) \mathcal{Q}^2 r^2. \quad (6.9)$$

All the symbols represent the same quantities as ours, and \mathcal{Q} is Melvin's charge parameter. We first begin by setting $D = 4$ and also choose $c = d$ to simplify some factors. Our solutions become

$$\text{Metric:} \quad ds^2 = -H(\rho)^{\frac{2}{1+2\alpha^2}} dt^2 + e^{-c\rho} H(\rho)^{-\frac{2}{1+2\alpha^2}} (e^{-c\rho} d\rho^2 + dx_1^2 + dx_2^2), \quad (6.10a)$$

$$\text{Scalar field:} \quad \varphi = -\frac{\alpha}{(1 + 2\alpha^2)} \ln H(\rho), \quad (6.10b)$$

$$\text{Gauge potential:} \quad A = -\frac{b \coth(b\rho)}{q(1 + 2\alpha^2)}, \quad (6.10c)$$

$$\text{Constraint:} \quad c = \frac{2b}{\sqrt{1 + 2\alpha^2}}, \quad (6.10d)$$

where $H(\rho) = \frac{q^2}{b^2} (1 + 2\alpha^2) \sinh^2(b\rho)$. We first note that the scalar fields have very similar forms which can be seen if we cast ours into Melvin's form in Eq. (6.9):

$$e^{-2\alpha\varphi} = \left(\sinh(b\rho) \sqrt{\frac{q^2(2\alpha^2 + 1)}{b^2}} \right)^{\frac{4\alpha^2}{2\alpha^2 + 1}}, \quad (6.11)$$

$$\Rightarrow L(r) = \sinh(b\rho) \sqrt{\frac{q^2(2\alpha^2 + 1)}{b^2}}, \quad (6.12)$$

where $\varphi_0 = 0$. Although this gives us a relationship between the r and ρ -coordinates, it is not a one-to-one mapping because the r -coordinate becomes undefined when $\rho \rightarrow 0$. As such, we were unable to transform our solution into the exact form in Eq. (6.8) using the above coordinate transformation. Nonetheless, we still see many similarities between our solution and Melvin's and thus, we will show that we can still recover a similar form using a different approach.

We begin by introducing the following coordinate transformations

$$\begin{aligned} t &\rightarrow ix_1, & x_1 &\rightarrow it, & \bar{r} &= \frac{2}{c} e^{-\frac{1}{2}c\rho}, \\ dt^2 &\rightarrow -dx_1^2, & dx_1^2 &\rightarrow -dt^2, & d\bar{r}^2 &= e^{-c\rho} d\rho^2. \end{aligned} \quad (6.13)$$

The first two transformations swaps one of the spatial coordinates with the temporal one to match Melvin's solution. The third transformation allows us to factor out the e^{-cp} in front of $d\rho^2$ in Eq. (6.10a). We may also rewrite the H -function as

$$\begin{aligned} H(r) &= \frac{2q}{c} \frac{1}{2} (e^{b\rho} - e^{-b\rho}) \\ &= \frac{q}{c} \left(\frac{cr}{2} \right)^{\sqrt{1+2\alpha^2}} \left(\left(\frac{2}{cr} \right)^{2\sqrt{1+2\alpha^2}} - 1 \right), \end{aligned} \quad (6.14)$$

where we have used the constraint in Eq. (6.10d) to substitute the constant b . Putting everything together, the new metric becomes

$$\begin{aligned} ds^2 &= \frac{c^2 r^2}{4} \left[\frac{2q}{c^2 r} \left(\frac{cr}{2} \right)^{\sqrt{1+2\alpha^2}} \left(\left(\frac{2}{cr} \right)^{2\sqrt{1+2\alpha^2}} - 1 \right) \right]^2 dx_1^2 \\ &\quad + \left[\frac{2q}{c^2 r} \left(\frac{cr}{2} \right)^{\sqrt{1+2\alpha^2}} \left(\left(\frac{2}{cr} \right)^{2\sqrt{1+2\alpha^2}} - 1 \right) \right]^{-2} (dr^2 - dt^2 + dx_2^2). \end{aligned} \quad (6.15)$$

Rescaling $x_1 \rightarrow \frac{2}{c}x_1$, we can write the metric into a Melvin-like form

$$ds^2 = r^2 K(r)^{-2} dx_1^2 + K(r)^2 (dr^2 - dt^2 + dx_2^2), \quad (6.16)$$

where

$$K(r) = \frac{c}{q} \left(\frac{2}{cr} \right)^{\sqrt{1+2\alpha^2}-1} \left(\left(\frac{2}{cr} \right)^{2\sqrt{1+2\alpha^2}} - 1 \right)^{-1}. \quad (6.17)$$

Although our metric has now been put into a Melvin-like form, it does not produce similar limiting results as Melvin's solution. For example, turning off the electric field in Melvin's solution is done by setting $\mathcal{Q} = 0 \Rightarrow L(r) = 1$ and the resulting metric simply becomes Minkowski spacetime in cylindrical coordinates.

Looking at our gauge potential solution in Eq. (6.10c), we have the relationship $A \propto \frac{1}{q}$. It is more intuitive to have a charge parameter which turns off the electric field when it is set to zero. We may do this by simply redefining $\frac{1}{q} \equiv Q$, then we have

$$K(r) = cQ \left(\frac{2}{cr} \right)^{\sqrt{1+2\alpha^2}-1} \left(\left(\frac{2}{cr} \right)^{2\sqrt{1+2\alpha^2}} - 1 \right)^{-1}. \quad (6.18)$$

We see that setting $Q = 0$ gives us $K(r) = 0$ and thus produces diverging terms in the metric as opposed to Melvin's solution. This is again, likely due to unfortunate coordinate choices. Nonetheless, we will show that in a specific limiting case, our solution may also reduce back to Minkowski spacetime.

Notice that if we choose $\alpha = 0$, the prefactor in Eq. (6.18) vanishes and we have

$$K(r) = cQ \left(\frac{c^2 r^2}{4 - c^2 r^2} \right). \quad (6.19)$$

Again, introducing the following coordinate transformations to absorb the charge parameter into the coordinates,

$$r \rightarrow \frac{r}{cQ}, \quad t \rightarrow \frac{t}{cQ}, \quad x_1 \rightarrow c^2 Q^2 x_1, \quad x_2 \rightarrow \frac{x_2}{cQ}, \quad (6.20)$$

our metric becomes

$$ds^2 = r^2 \bar{K}(r)^{-2} dx_1^2 + \bar{K}(r)^2 (dr^2 - dt^2 + dx_2^2), \quad (6.21)$$

with

$$\bar{K}(r) = \frac{1}{\frac{4Q^2}{r^2} - 1}. \quad (6.22)$$

Thus, in the limit where $Q \rightarrow 0$,

$$\lim_{Q \rightarrow 0} \bar{K}^2 = 1. \quad (6.23)$$

Therefore, we have successfully found a limiting case for $\Lambda = 0$ whereby our solution reduces back to Minkowski spacetime as expected in the absence of an electric field. While this should technically also be true for any value of α , we are actually free to choose any α since turning off the EM fields corresponds to $F^2 = 0$ in the EMD action in Eq. (2.1) and therefore regardless of the coupling constant, the EM term still vanishes.

6.2 Geodesics

In this section, we will be investigating the geodesic structures of both solutions. A geodesic may be described by a trajectory $x^\mu(\tau)$, where τ is some parameter along

the curve. The invariant Lagrangian for geodesic motion is given by [27]

$$L = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \frac{\epsilon}{2}, \quad (6.24)$$

where

$$\epsilon = \begin{cases} -1, & \text{For time-like geodesics} \\ 0, & \text{For null/photon geodesics} \\ +1, & \text{For space-like geodesics} \end{cases} \quad (6.25)$$

The above may also be familiarly known as the normalization of 4-velocity in (3+1)-dimensions. Applying the Euler-Lagrange equation on Eq. (6.24) gives rise to the geodesic equation which is known in the following two equivalent forms

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma_{\mu\nu}^\alpha \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0 \quad \text{or} \quad \frac{d}{d\tau} \left(g_{\alpha\mu} \frac{dx^\mu}{d\tau} \right) - \frac{1}{2} (\partial_\alpha g_{\mu\nu}) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0. \quad (6.26)$$

The $(D-1)$ Killing vectors in our metric ansatz gives rise to $(D-1)$ conserved quantities, which can easily be seen by deriving the geodesic equations for the coordinates t and x_i :

$$\begin{aligned} g_{tt} \frac{dt}{d\tau} &= E & g_{ii} \frac{dx^i}{d\tau} &= P_i \\ \Rightarrow \frac{dt}{d\tau} &= -E e^{-2F_0}, & \Rightarrow \frac{dx^i}{d\tau} &= P_i e^{-2F_i}, \end{aligned} \quad (6.27) \quad (6.28)$$

where E and the P_i 's are quantities which are conserved along all geodesic trajectories in our spacetime, arising from the direct integration of Eq. (6.26). These may be interpreted as the energy and x_i -direction momenta of the test particle respectively. The geodesic equation for the ρ -coordinate is naturally more complicated and is difficult to interpret. We thus obtain an equivalent equation by plugging in Eq. (6.27) and Eq. (6.28) directly into the Lagrangian in Eq. (6.24) which gives rise to the first order equation

$$\begin{aligned} -E^2 e^{-2F_0} + e^{2H} \left(\frac{d\rho}{d\tau} \right)^2 + \sum_{i=1}^n (P_i^2 e^{-2F_i}) &= \epsilon \\ \Rightarrow \left(\frac{d\rho}{d\tau} \right)^2 &= e^{-2H-2F_0} (E^2 - V_{\text{eff}}^2), \end{aligned} \quad (6.29)$$

where we have defined the effective potential to be

$$V_{\text{eff}}^2 = \epsilon e^{2F_0} + \sum_{i=1}^n (P_i^2 e^{2F_0 - 2F_i}) \quad (6.30)$$

For simplicity, we shall consider Cases 1 and 2 in 4-dimensions in the following section. We will also only consider time-like geodesics ($\epsilon = -1$), because the first term often becomes a multiple of an exponential function and a hyperbolic function and thus tuning the parameters allows one to dominate the other, resulting in interesting features in the graphs. The null geodesic ($\epsilon = 0$) effective potentials are not as rich in features without the first term.

6.2.1 Effective Potentials of Case 1: $q = 0$

We have previously discussed two special situations in this case at the beginning of this chapter – setting the arbitrary constants to $c = d$ (non-zero scalar field) and $c = -d$ (zero scalar field). The constraint equations from Eq. (4.20) for both these choices reduce to

$$c = d : \quad c^2 = \frac{4b^2(\beta - 2\alpha)^2}{(2\beta^2 - 3)(8 + 5\beta^2)}, \quad (6.31)$$

$$c = -d : \quad c^2 = \frac{4b^2}{3(2\beta^2 - 3)}. \quad (6.32)$$

We see that we have the requirement $2\beta^2 > 3$ to ensure a positive value for c^2 . This is equivalent to $\beta^2 > \frac{D-1}{D-2}$ for $D = 4$. Therefore, choosing a positive cosmological constant would result in the sinh function in our metric whereas a negative cosmological constant would produce a cosh function instead, as discussed in Chapter 4. The effective potentials are then obtained by substituting our solutions into Eq. (6.30) and are found to be

$$V_{\text{eff}, q=0}^2 \Big|_{c=d} = e^{-\frac{4b\beta\rho}{\sqrt{(2\beta^2-3)(8+5\beta^2)}}} G(\rho)^{\frac{1}{2\beta^2-3}} + \sum_{i=1}^2 \left(P_i^2 e^{-\frac{6b\beta\rho}{\sqrt{(2\beta^2-3)(8+5\beta^2)}}} \right), \quad (6.33)$$

$$V_{\text{eff}, q=0}^2 \Big|_{c=-d} = e^{\frac{4b\rho}{\sqrt{3(2\beta^2-3)}}} G(\rho)^{\frac{1}{2\beta^2-3}} + \sum_{i=1}^n \left(P_i^2 e^{-\frac{6b\rho}{\sqrt{3(2\beta^2-3)}}} \right), \quad (6.34)$$

where

$$G(\rho) = \begin{cases} \frac{\Lambda}{b^2} (2\beta^2 - 3) \sinh^2(b\rho), & \text{for } \Lambda > 0, \\ -\frac{\Lambda}{b^2} (2\beta^2 - 3) \cosh^2(b\rho), & \text{for } \Lambda < 0. \end{cases} \quad (6.35)$$

6.2.2 Effective Potential of Case 2: $\Lambda = 0$

In the same way, we also first look at the constraint from Eq. (4.25) for Case 2 where we also simplify with $c = d$ and $c = -d$ (here there is no physical difference between the choices, it is purely a mathematical simplification):

$$c = d: \quad c^2 = \frac{4b^2}{(2\alpha^2 + 1)}, \quad (6.36)$$

$$c = -d: \quad c^2 = \frac{4b^2(\beta - 2\alpha)^2}{(2\alpha^2 + 1)(8\alpha\beta + \beta^2 - 8)}. \quad (6.37)$$

We expect that if $\Lambda = 0$, then the Liouville coupling constant β should either drop off or cancel out in the metric solution. As such, we see from the above that $c = -d$ is not a valid choice since we still have a dependence on β . Therefore we shall only consider the $c = d$ case. For $\Lambda = 0$, the $H(\rho)$ function has a prefactor of q^2 which cannot take on negative values. Hence we only need to consider one possible effective potential

$$V_{\text{eff}, \Lambda=0}^2 = H(\rho)^{\frac{1}{1+2\alpha^2}} + \sum_{i=1}^2 \left(P_i^2 e^{c_2 \rho} H(\rho)^{\frac{2}{(1+2\alpha^2)}} \right), \quad (6.38)$$

where

$$H(\rho) = \frac{q^2}{b^2} (1 + 2\alpha^2) \sinh^2(b\rho). \quad (6.39)$$

6.2.3 Plots of the Effective Potentials

Here, we present the graphical plots of the effective potentials for both solutions. For each plot, only one constant will be varied to show how it affects the general shape of the effective potential. The constants we are interested in varying will be the Liouville coupling constants α and β , and b . All other constants are merely scaling factors and do not change the shape of the graph. As mentioned before, α and β will only take on positive values, whereas b is free to take on any real value (or even imaginary, which we will only discuss about in Chapter 7). There are many possible numerical combinations given the number of free parameters, but we shall only present a handful which we have found to display interesting features.

Firstly, note that we have scaled some of the graphs differently along the ρ -axis to zoom into the main features of the graph. We remind the reader that in our coordinate, $\rho \rightarrow -\infty$ approaches the origin and $\rho = 0$ is far away from the origin. We are generally interested in features such as potential wells, potential barriers and maximas/minimas in potential plots. Potential wells depict bounded states, potential barriers present regions which are not accessible by the test particles, and maximas/minimas represent unstable/stable equilibrium orbits respectively.

We first take a look at Figure 6.1 and Figure 6.2 which have opposite signs for the cosmological constant with the existence of a scalar field. We have found that for a very specific range of $1.57 \leq \beta \leq 1.59$, the potential graph transits from a repulsive source (pushing away from the origin) to an attractive source (pulling in towards the origin) as β increases. This is due to the competition between the exponential function which is decaying with a negative argument, and the sinh and cosh functions which diverge for large ρ . As a consequence, the existence of a stable orbit also vanishes in both graphs. For Figure 6.1 with the negative Λ , there is an infinite potential barrier at $\rho=0$ whereas Figure 6.2 with the positive Λ possesses another point of unstable equilibrium near $\rho = 0$.

Figure 6.3 and Figure 6.4 shows how the sign of b affects the shape of the effective potential as it approaches the origin. Positive values depict a repulsive system, whilst negative values changes the system to become an attractive one.

In the absence of a scalar field, the curves in Figure 6.5, Figure 6.6, Figure 6.7 and Figure 6.8 all approach infinity as $\rho \rightarrow -\infty$. For a negative Λ , this is in agreement with the well-known AdS case [28]. In Figure 6.5 and Figure 6.6, we also see that there exists a point of stable equilibrium which disappears as β increases from the minimum value of $\beta^2 > 3$.

The last three plots in Figure 6.9, Figure 6.10 and Figure 6.11 are for Case 2 with $\Lambda = 0$. We see that when b takes on a positive value in Figure 6.10, decreasing α leads to the disappearance of the unstable equilibrium and also changes the system from an attractive one to a repulsive one.

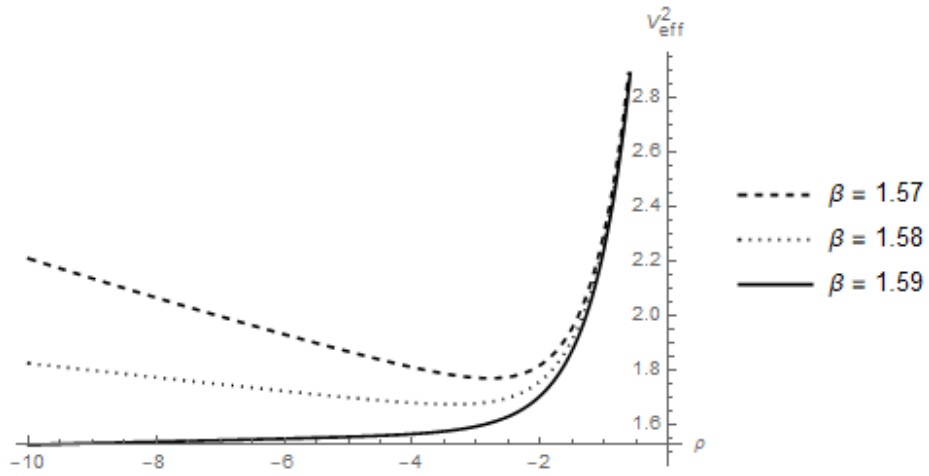


Figure 6.1: Case 1: $q = 0$, *non-zero* scalar field; $P_1 = P_2 = 1$, $b = -1$, $\Lambda = -5$.

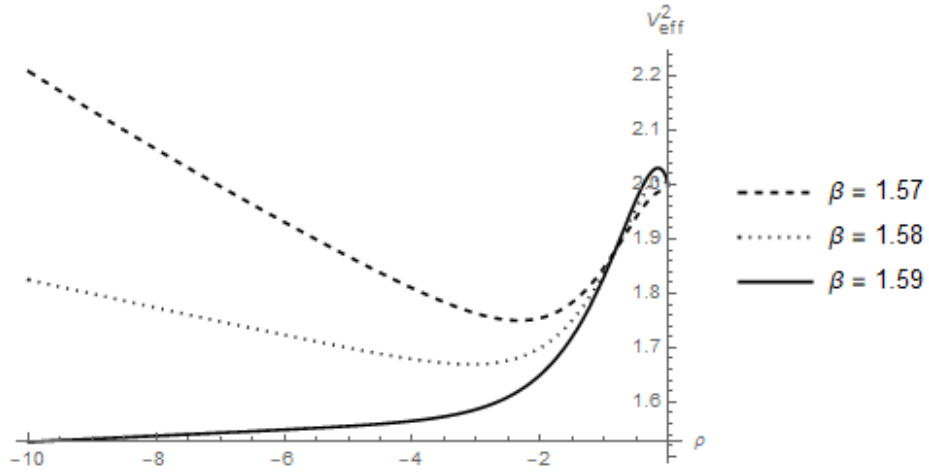


Figure 6.2: Case 1: $q = 0$, *non-zero* scalar field; $P_1 = P_2 = 1$, $b = -1$, $\Lambda = +5$.

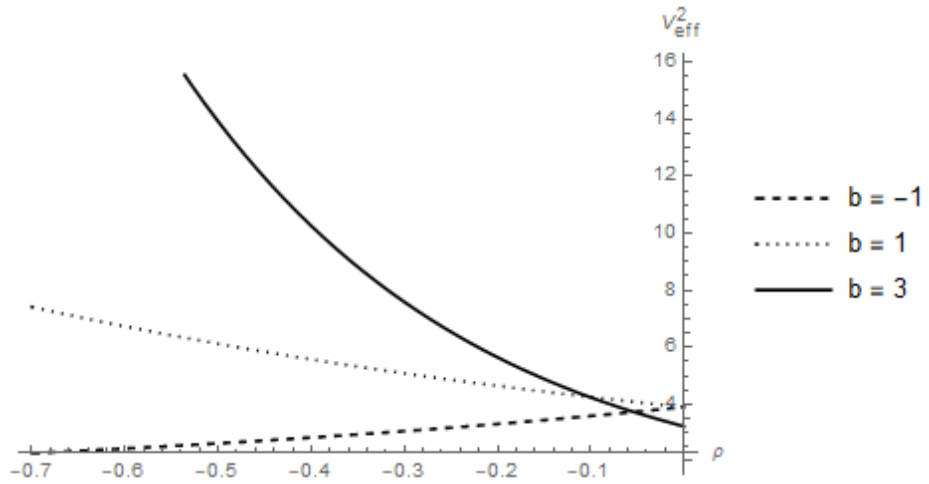


Figure 6.3: Case 1: $q = 0$, *non-zero* scalar field; $P_1 = P_2 = 1$, $\beta = 2$, $\Lambda = -5$.

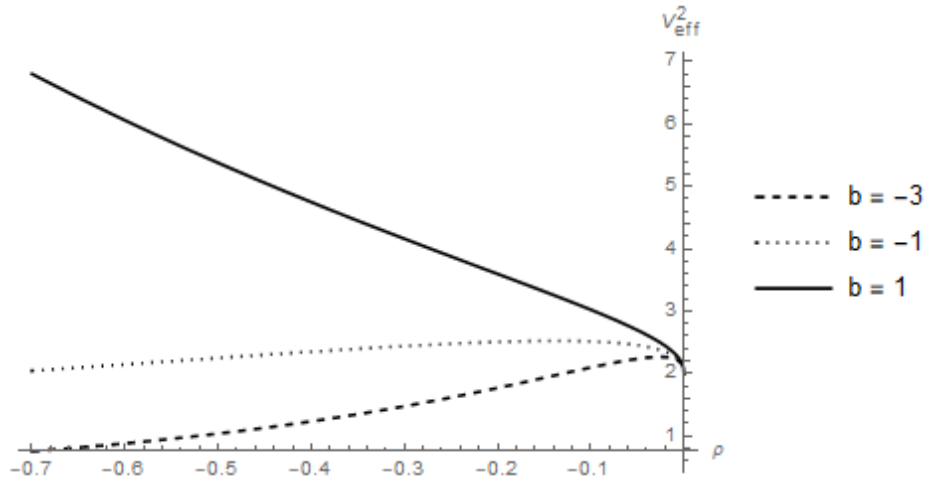


Figure 6.4: Case 1: $q = 0$, *non-zero* scalar field; $P_1 = P_2 = 1$, $\beta = 2$, $\Lambda = +5$.

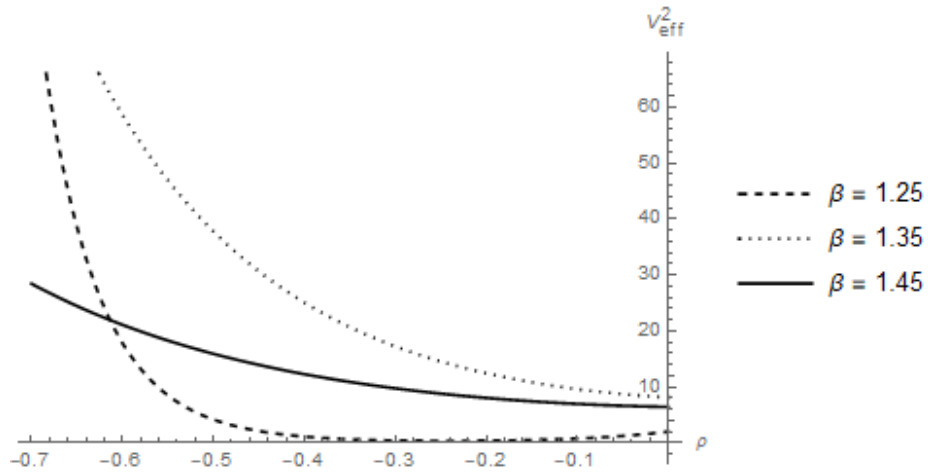


Figure 6.5: Case 1: $q = 0$, *zero* scalar field; $P_1 = P_2 = 1$, $b = -1$, $\Lambda = -5$.

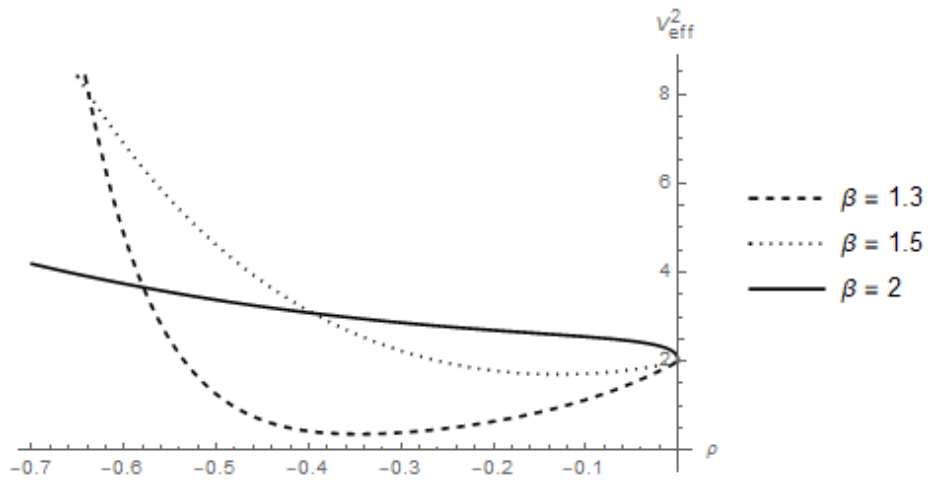


Figure 6.6: Case 1: $q = 0$, *zero* scalar field; $P_1 = P_2 = 1$, $b = -1$, $\Lambda = +5$.

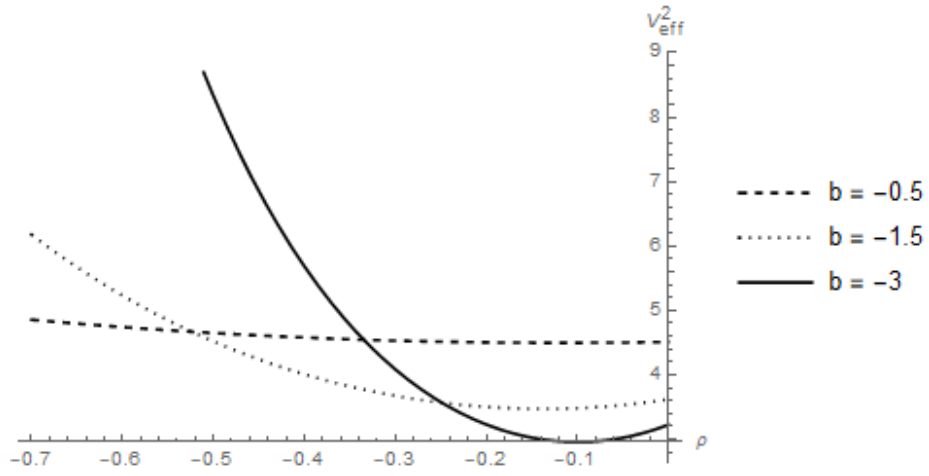


Figure 6.7: Case 1: $q = 0$, zero scalar field; $P_1 = P_2 = 1$, $\beta = 2$, $\Lambda = -5$.

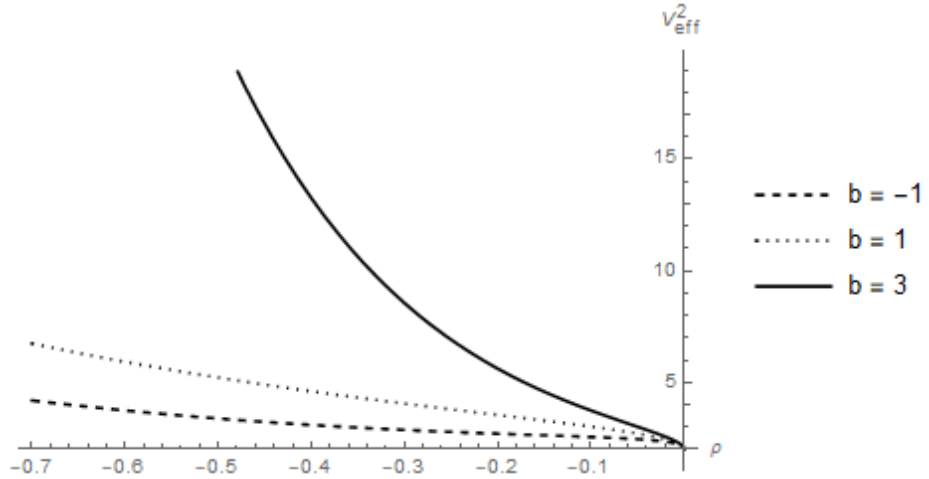


Figure 6.8: Case 1: $q = 0$, zero scalar field; $P_1 = P_2 = 1$, $\beta = 2$, $\Lambda = +5$.

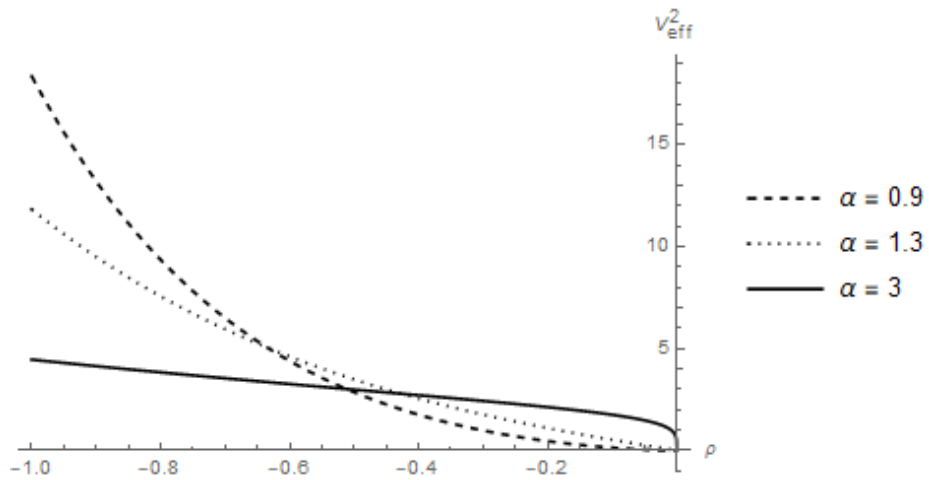
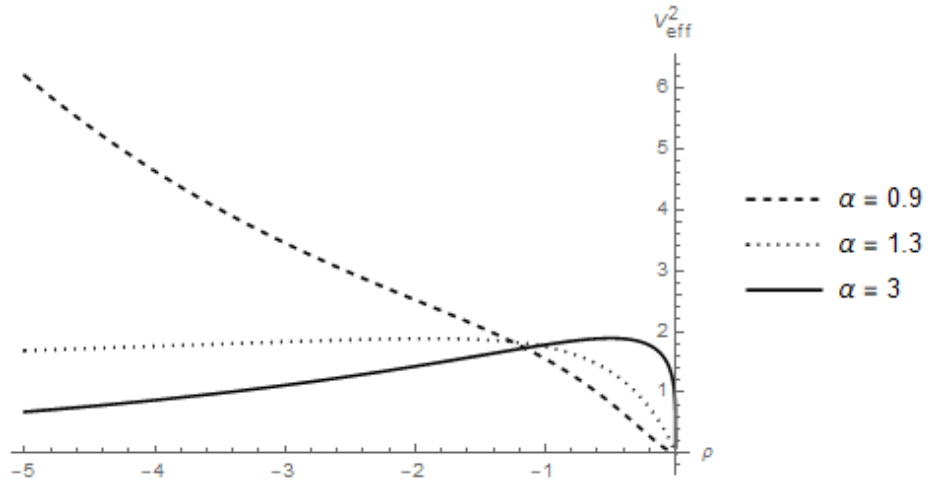
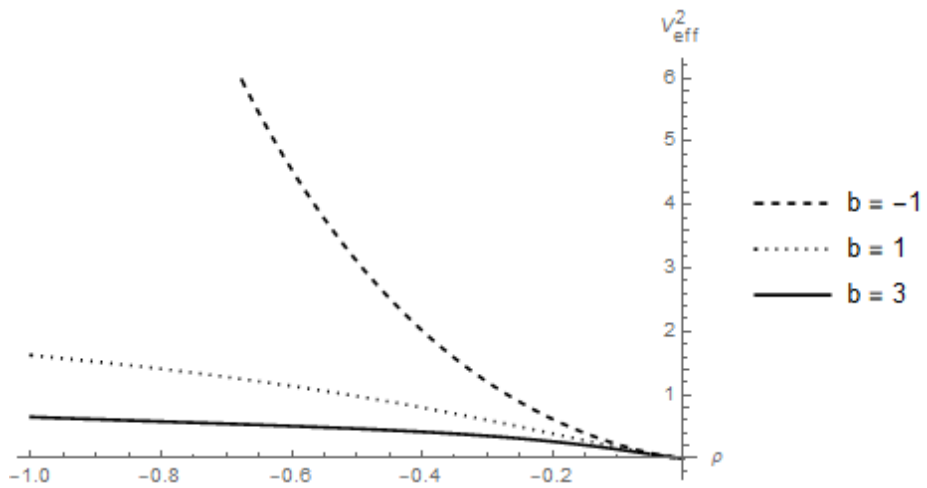


Figure 6.9: Case 2: $\Lambda = 0$; $P_1 = P_2 = 1$, $b = -1$, $q = 1$.

Figure 6.10: Case 2: $\Lambda = 0$; $P_1 = P_2 = 1$, $b = 1$, $q = 1$.Figure 6.11: Case 2: $\Lambda = 0$; $P_1 = P_2 = 1$, $\alpha = 1$, $q = 1$.

Chapter 7

Conclusion

7.1 Summary

The main focus of this project is to investigate solutions in EMD gravity and to also search for new solutions in EMD gravity. As mentioned in the introduction, we have explained how the AdS/CFT correspondence has sparked much research in the search for solutions in general relativity and therefore motivated us to undertake this project.

We began by showing in Chapter 3 how Lim's solution for Einstein-Dilaton gravity was linked to Ren's solution in pure Einstein gravity by turning off the scalar field. Mathematically, we showed how this was closely tied to Kasner's conditions which accompanied Ren's metric.

Following which, we derived an entire class of solutions in EMD gravity under a specific set of assumptions introduced in Chapter 4 which was inspired by Maki and explained the motivation behind each assumption. We have extended her methods to arbitrary dimensions and showed that there were only four possible ways to decouple the field equations such that we could solve them using the Liouville differential equation. Within each solution case, some were further split into two special cases, depending on the overall signs of the prefactors.

Chapter 5 provides insight on how we could interpret our coordinate choice by making a comparison between our scalar field solution for Case 1 to Lim's scalar field in Einstein-Dilaton gravity. We obtained a relationship between our coordinates and his,

which allowed for a qualitative deduction for the singularities present in our solutions. We then showed that Cases 1 and 2 were in agreement with the Cosmic Censorship Hypothesis and are therefore black hole solutions. We also briefly examined Cases 3 and 4 for simple choices of the arbitrary constants and showed that their curvatures were non-trivial.

In Chapter 6, we explained that we were unable to find an appropriate coordinate transformation which could map our solution back to Lim's due to the incomplete mapping between our ρ -coordinate and Lim's r -coordinate. Nevertheless, we had obtained a solution for a more general case involving a Liouville coupling to the cosmological constant which reproduced the correct curvature when reducing it back to simpler systems. We also successfully showed that our solution for Case 2 resembled the Melvin solution and were able to express it in an identical form. Again, an incomplete mapping of coordinates caused us to be unsuccessful in obtaining an exact coordinate transformation. However, by rescaling our coordinates, we were able to show that our solution reduces back to Minkowski spacetime in the limit $Q \rightarrow 0$, which was in agreement with the Melvin solution.

We also investigated the geodesic structure of both cases and presented the plots of the effective potentials. We discovered various interesting features such as the particular range of $1.57 \leq \beta \leq 1.59$ for Case 1 in the presence of a scalar field which produced a transition from a repulsive system to an attractive one. A similar situation occurs when changing the sign of b in the same case. We also see this happening in Case 2 with a positive b , where decreasing the value of α changes the vanishing effective potential at $\rho \rightarrow -\infty$ to a diverging one.

7.2 Future Work

7.2.1 Trigonometric Functions

In our derivation of our solutions in Chapter 4, we used the hyperbolic functions as the solutions to the Liouville equation. The arbitrary constant b arising from integration is not required to be real. As such, suppose we chose $b = i\bar{b}$ such that $b^2 = -\bar{b}^2$, recalling the relationship between the hyperbolic and trigonometric functions:

$$\sinh(ix) = i \sin(x),$$

$$\cosh(ix) = \cos(x),$$

the Liouville differential equation solutions could then be rewritten as

$$\text{Positive } k: \phi_+(\rho) = -\ln\left(\frac{\sqrt{k}}{\bar{b}} \cos(\bar{b}\rho)\right), \quad (7.1)$$

$$\text{Negative } k: \phi_-(\rho) = -\ln\left(i\frac{\sqrt{-k}}{\bar{b}} \sin(\bar{b}\rho)\right). \quad (7.2)$$

Now obviously Eq. (7.2) is completely unphysical with an imaginary argument. However, Eq. (7.1) is a perfectly legitimate solution which could exhibit totally different features from the associated sinh solution for positive k .

7.2.2 Coordinate Choice

As evident in many of our chapters, our choice of coordinates happens to be very difficult to map back to known coordinate systems despite of the similar forms of our solutions when comparing to other authors. This appears to boil back to our metric ansatz being expressed in exponential functions. While this drastically simplified earlier calculations, it turned out to make it difficult for us to find coordinate transformations to map our solutions to others even though they often seem highly likely to be the same. This appears to come from the inversions of the exponential functions producing logarithmic functions which do not behave very well when the arguments become negative.

A different choice of metric ansatz, perhaps a power series or Fourier series, could possibly lead to newer insights into EMD gravity. In addition, we could also change the coupling functions to match with the metric ansatz since that was how we managed to solve our differential equations.

Appendix A

Variation of EMD Action

We begin with the action

$$S = \frac{1}{2} \int d^D x \sqrt{-g} (R - 2\Lambda e^{2\beta\varphi} - (\nabla\varphi)^2 - e^{-2\alpha\varphi} F^2). \quad (\text{A.1})$$

Varying the action gives us

$$\begin{aligned} \delta S = & \frac{1}{2} \int d^D x \delta (\sqrt{-g}) (R - 2\Lambda e^{2\beta\varphi} - (\nabla\varphi)^2 - e^{-2\alpha\varphi} F^2) \\ & + \frac{1}{2} \int d^D x \sqrt{-g} \left(\delta R - \delta (2\Lambda e^{2\beta\varphi}) - \delta (g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi) - \delta (e^{-2\alpha\varphi} F^2) \right). \end{aligned} \quad (\text{A.2})$$

A.1 Varying Each Term

A.1.1 Metric Determinant

We first begin with Jacobi's formula [29] for differentiating a determinant

$$\delta(g) = g g^{\mu\nu} \delta g_{\mu\nu}. \quad (\text{A.3})$$

Then we have

$$\begin{aligned} \delta (\sqrt{-g}) &= -\frac{1}{2} \frac{1}{\sqrt{-g}} \delta(g) \\ &= -\frac{1}{2} \sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu} \\ &= -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}. \end{aligned} \quad (\text{A.4})$$

A.1.2 Ricci Scalar

The variation is

$$\begin{aligned}
\delta(R) &= \delta(g^{\mu\nu} R_{\mu\nu}) \\
&= R_{\mu\nu} \delta g^{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu} \\
&= R_{\mu\nu} \delta g^{\mu\nu} + g^{\mu\nu} \left(\nabla_\rho (\delta\Gamma_{\nu\mu}^\rho) - \nabla_\nu (\delta\Gamma_{\sigma\mu}^\sigma) \right) \\
&= R_{\mu\nu} \delta g^{\mu\nu} + \underbrace{\nabla_\rho (g^{\mu\nu} \delta\Gamma_{\nu\mu}^\rho - g^{\mu\rho} \delta\Gamma_{\sigma\mu}^\sigma)}_{\text{total derivative}}, \tag{A.5}
\end{aligned}$$

where we used the Palatini identity [30] in the third step. For the last step, we used the metric compatibility of the covariant derivative, $\nabla_\rho g^{\mu\nu} = 0$, and also swapped dummy indices $\nu \rightarrow \rho$ for the last term so that we may factor out covariant derivative.

A.1.3 Coupled Cosmological Constant

The variation of the coupled cosmological constant is simply

$$\delta(2\Lambda e^{2\beta\varphi}) = 4\Lambda\beta e^{2\beta\varphi} \delta\varphi. \tag{A.6}$$

A.1.4 Scalar Field

The variation is

$$\begin{aligned}
\delta(g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi) &= (\nabla_\mu \varphi \nabla_\nu \varphi) \delta g^{\mu\nu} + g^{\mu\nu} \nabla_\mu \delta\varphi \nabla_\nu \varphi + g^{\mu\nu} \nabla_\nu \delta\varphi \nabla_\mu \varphi \\
&= (\nabla_\mu \varphi \nabla_\nu \varphi) \delta g^{\mu\nu} + 2\nabla^\mu \delta\varphi \nabla_\mu \varphi \\
&= (\nabla_\mu \varphi \nabla_\nu \varphi) \delta g^{\mu\nu} + \underbrace{2\nabla^\mu (\delta\varphi \nabla_\mu \varphi)}_{\text{total derivative}} - 2(\nabla^\mu \nabla_\mu \varphi) \delta\varphi, \tag{A.7}
\end{aligned}$$

where we used integration by parts in the last step.

A.1.5 Coupled EM Tensor

We first apply the product rule to get

$$\delta(e^{-2\alpha\varphi} F^2) = -2\alpha e^{-2\alpha\varphi} F^2 \delta\varphi + e^{-2\alpha\varphi} \delta F^2. \tag{A.8}$$

To proceed, we recall that

$$\begin{aligned} F^2 &= F^{\mu\nu} F_{\mu\nu} \\ &= g^{\mu\sigma} g^{\nu\beta} F_{\sigma\beta} F_{\mu\nu}. \end{aligned} \quad (\text{A.9})$$

We also have the definition for the EM tensor

$$\begin{aligned} F_{\mu\nu} &= \nabla_\mu A_\nu - \nabla_\nu A_\mu \\ &= \partial_\mu A_\nu - \cancel{\Gamma_{\mu\nu}^\sigma A_\sigma} - \partial_\nu A_\mu + \cancel{\Gamma_{\nu\mu}^\sigma A_\sigma} \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu, \end{aligned} \quad (\text{A.10})$$

where the Christoffel symbols cancel due to symmetries in the lower indices. The final form of the EM tensor is the most commonly used, but we bear in mind that we have to keep the covariant derivative notation in the following steps. Therefore we have

$$\begin{aligned} \delta F^2 &= g^{\nu\beta} F_{\sigma\beta} F_{\mu\nu} \delta g^{\mu\sigma} + g^{\mu\sigma} F_{\sigma\beta} F_{\mu\nu} \delta g^{\nu\beta} + g^{\mu\sigma} g^{\nu\beta} F_{\mu\nu} \delta F_{\sigma\beta} + g^{\mu\sigma} g^{\nu\beta} F_{\sigma\beta} \delta F_{\mu\nu} \\ &= 2g^{\nu\beta} F_{\sigma\beta} F_{\mu\nu} \delta g^{\mu\sigma} + 2g^{\mu\sigma} g^{\nu\beta} F_{\mu\nu} \delta F_{\sigma\beta} \\ &= 2F_{\sigma\beta} F_\mu^\beta \delta g^{\mu\sigma} + 2F^{\sigma\beta} \delta F_{\sigma\beta}. \end{aligned} \quad (\text{A.11})$$

The second term in Eq. (A.11) may be rewritten as

$$\begin{aligned} 2F^{\sigma\beta} \delta F_{\sigma\beta} &= 2F^{\sigma\beta} \delta(\nabla_\sigma A_\beta - \nabla_\beta A_\sigma) \\ &= 2F^{\sigma\beta} (\nabla_\sigma \delta A_\beta - \nabla_\beta \delta A_\sigma) \end{aligned} \quad (\text{A.12})$$

$$\begin{aligned} &\downarrow \text{Since } F_{\sigma\beta} \text{ is anti-symmetric, } -\nabla_\beta \delta A_\sigma = \nabla_\sigma \delta A_\beta \\ &= 4F^{\sigma\beta} \nabla_\sigma \delta A_\beta. \end{aligned} \quad (\text{A.13})$$

Putting everything back into Eq. (A.8), we get

$$\begin{aligned} \delta(e^{-2\alpha\varphi} F^2) &= -2\alpha e^{-2\alpha\varphi} F^2 \delta\varphi + e^{-2\alpha\varphi} \left(2F_{\sigma\beta} F_\mu^\beta \delta g^{\mu\sigma} + 4F^{\sigma\beta} \nabla_\sigma \delta A_\beta \right) \\ &= -2\alpha e^{-2\alpha\varphi} F^2 \delta\varphi + 2e^{-2\alpha\varphi} F_{\nu\beta} F_\mu^\beta \delta g^{\mu\nu} - 4(\nabla_\sigma e^{-2\alpha\varphi} F^{\sigma\beta}) \delta A_\beta \\ &\quad + \underbrace{4\nabla_\sigma (e^{-2\alpha\varphi} F^{\sigma\beta} \delta A_\beta)}_{\text{Full derivative}}. \end{aligned} \quad (\text{A.14})$$

A.2 Final Variation of the EMD Action

The total derivative terms in a few of the expressions above are boundary terms when integrated. The variation of each field variable is only done within a neighbourhood of the boundary and therefore the boundary terms do not contribute to the variation of the action. Hence, we may drop all total derivative terms. We obtain the desired variation after substituting everything back into Eq. (A.2):

$$\begin{aligned}
\delta S = & \frac{1}{2} \int d^D x \sqrt{-g} \left[\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda e^{2\beta\varphi} g_{\mu\nu} + \frac{1}{2} g_{\mu\nu} (\nabla\varphi)^2 - \nabla_\mu\varphi \nabla_\nu\varphi \right. \right. \\
& \left. \left. - 2e^{-2\alpha\varphi} F_{\nu\lambda} F_\mu^\lambda + \frac{1}{2} e^{-2\alpha\varphi} F^2 g_{\mu\nu} \right) \delta g^{\mu\nu} - 2 \left((\nabla^2\varphi) + \alpha e^{-2\alpha\varphi} F^2 - 2\Lambda\beta e^{2\beta\varphi} \right) \delta\varphi \right. \\
& \left. - 4 \left(\nabla_\sigma e^{-2\alpha\varphi} F^{\sigma\lambda} \right) \delta A_\lambda \right]. \tag{A.15}
\end{aligned}$$

Appendix B

Calculation of Christoffel Symbols

Our metric ansatz is given as

$$ds^2 = -e^{2F_0(\rho)} dt^2 + e^{2H(\rho)} d\rho^2 + \sum_{i=1}^n e^{2F_i(\rho)} dx_i^2. \quad (\text{B.1})$$

The Christoffel symbols may be calculated from

$$\Gamma_{\mu\nu}^{\kappa} = \frac{1}{2} g^{\kappa\lambda} (\partial_{\mu} g_{\lambda\nu} + \partial_{\nu} g_{\lambda\mu} - \partial_{\lambda} g_{\mu\nu}). \quad (\text{B.2})$$

In the absence of torsion, the Christoffel symbols are symmetric, ie.

$$\Gamma_{ij}^{\kappa} = \Gamma_{ji}^{\kappa}. \quad (\text{B.3})$$

Even though we are dealing with $n + 2$ dimensions, all the g_{ii} components are essentially of the same form. Therefore, rather than explicitly calculating the Christoffel symbols with all n -terms, we shall simply reduce the dimensions to four and call $g_{ii} = e^{2F_i}$ and $g_{jj} = e^{2F_j}$ with $i \neq j$. Thus we are only required to find the following components:

$$\begin{pmatrix} \Gamma_{tt}^{\kappa} & \Gamma_{ti}^{\kappa} & \Gamma_{tj}^{\kappa} & \Gamma_{t\rho}^{\kappa} \\ & \Gamma_{ii}^{\kappa} & \Gamma_{ij}^{\kappa} & \Gamma_{i\rho}^{\kappa} \\ & & \Gamma_{jj}^{\kappa} & \Gamma_{j\rho}^{\kappa} \\ & & & \Gamma_{\rho\rho}^{\kappa} \end{pmatrix}. \quad (\text{B.4})$$

Since the metric is diagonal and only dependent on the ρ -coordinate, we immediately have

$$\Gamma_{ti}^{\kappa} = \Gamma_{tj}^{\kappa} = \Gamma_{ij}^{\kappa} = 0. \quad (\text{B.5})$$

Explicitly calculating all other Christoffel symbols, we get:

$$\begin{aligned}
\Gamma_{tt}^{\kappa} &= \frac{1}{2}g^{\kappa\lambda}(\partial_t g_{\lambda t} + \partial_t g_{\lambda t} - \partial_{\lambda} g_{tt}) \\
&= \frac{1}{2}g^{\kappa\rho}[-\partial_{\rho}(-e^{2F_0})] \\
&= g^{\kappa\rho}e^{2F_0}F_0',
\end{aligned} \tag{B.6}$$

$$\begin{aligned}
\Gamma_{ii}^{\kappa} &= \frac{1}{2}g^{\kappa\lambda}(\partial_i g_{\lambda i} + \partial_i g_{\lambda i} - \partial_{\lambda} g_{ii}) \\
&= \frac{1}{2}g^{\kappa\rho}[-\partial_{\rho}(e^{2F_i})] \\
&= -g^{\kappa\rho}e^{2F_i}F_i',
\end{aligned} \tag{B.7}$$

$$\begin{aligned}
\Gamma_{jj}^{\kappa} &= \frac{1}{2}g^{\kappa\lambda}(\partial_j g_{\lambda j} + \partial_j g_{\lambda j} - \partial_{\lambda} g_{jj}) \\
&= \frac{1}{2}g^{\kappa\rho}[-\partial_{\rho}(e^{2F_j})] \\
&= -g^{\kappa\rho}e^{2F_j}F_j',
\end{aligned} \tag{B.8}$$

$$\begin{aligned}
\Gamma_{\rho\rho}^{\kappa} &= \frac{1}{2}g^{\kappa\lambda}(\partial_{\rho} g_{\lambda\rho} + \partial_{\rho} g_{\lambda\rho} - \partial_{\lambda} g_{\rho\rho}) \\
&= \frac{1}{2}g^{\kappa\rho}[\partial_{\rho}(e^{2H})] \\
&= g^{\kappa\rho}e^{2H}H',
\end{aligned} \tag{B.9}$$

$$\begin{aligned}
\Gamma_{t\rho}^{\kappa} &= \frac{1}{2}g^{\kappa\lambda}(\partial_t g_{\lambda\rho} + \partial_{\rho} g_{\lambda t} - \partial_{\lambda} g_{t\rho}) \\
&= \frac{1}{2}g^{\kappa t}[\partial_{\rho}(-e^{2F_0})] \\
&= -g^{\kappa t}e^{2F_0}F_0',
\end{aligned} \tag{B.10}$$

$$\begin{aligned}
\Gamma_{i\rho}^{\kappa} &= \frac{1}{2}g^{\kappa\lambda}(\partial_i g_{\lambda\rho} + \partial_{\rho} g_{\lambda i} - \partial_{\lambda} g_{i\rho}) \\
&= \frac{1}{2}g^{\kappa i}[\partial_{\rho}(e^{2F_i})] \\
&= g^{\kappa i}e^{2F_i}F_i',
\end{aligned} \tag{B.11}$$

$$\begin{aligned}
\Gamma_{j\rho}^{\kappa} &= \frac{1}{2}g^{\kappa\lambda}(\partial_j g_{\lambda\rho} + \partial_{\rho} g_{\lambda j} - \partial_{\lambda} g_{j\rho}) \\
&= \frac{1}{2}g^{\kappa j}[\partial_{\rho}(e^{2F_j})] \\
&= g^{\kappa j}e^{2F_j}F_j'.
\end{aligned} \tag{B.12}$$

From Eq. (B.7), Eq. (B.8), Eq. (B.11) and Eq. (B.12), it is clear that the Christoffel symbols involving the x_i and x_j coordinates are the same. Therefore, generalizing

back to D -dimensions, the non-zero Christoffel symbols are

$$\Gamma_{tt}^\rho = e^{2F_0-2H} F_0', \quad (\text{B.13a})$$

$$\Gamma_{ii}^\rho = -e^{2F_i-2H} F_i', \quad (\text{B.13b})$$

$$\Gamma_{\rho\rho}^\rho = H', \quad (\text{B.13c})$$

$$\Gamma_{t\rho}^t = \Gamma_{\rho t}^t = F_0', \quad (\text{B.13d})$$

$$\Gamma_{i\rho}^i = \Gamma_{\rho i}^i = F_i', \quad (\text{B.13e})$$

where $i = 1, 2, \dots, n$.

Appendix C

Calculating Ricci Tensor Components

The Ricci tensor is given by

$$R_{\sigma\nu} = \partial_\mu \Gamma_{\nu\sigma}^\mu - \partial_\nu \Gamma_{\mu\sigma}^\mu + \Gamma_{\mu\lambda}^\mu \Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\lambda}^\mu \Gamma_{\mu\sigma}^\lambda. \quad (\text{C.1})$$

We are only interested in the diagonal components since our metric ansatz is diagonal. Before proceeding, we may again simplify the problem by considering the four dimensional case again. Thus, we get the following components:

$$\begin{aligned} R_{tt} &= \partial_\mu \Gamma_{tt}^\mu - \cancel{\partial_t \Gamma_{\mu t}^\mu} + \Gamma_{\mu\lambda}^\mu \Gamma_{tt}^\lambda - \Gamma_{t\lambda}^\mu \Gamma_{\mu t}^\lambda \\ &= \partial_\rho \Gamma_{tt}^\rho + \Gamma_{i\rho}^i \Gamma_{tt}^\rho + \Gamma_{j\rho}^j \Gamma_{tt}^\rho + \Gamma_{\rho\rho}^\rho \Gamma_{tt}^\rho + \cancel{\Gamma_{t\rho}^t \Gamma_{tt}^\rho} - \Gamma_{tt}^\rho \Gamma_{\rho t}^t - \cancel{\Gamma_{t\rho}^t \Gamma_{tt}^\rho} \\ &= [e^{2F_0-2H} (2F_0' - 2H') F_0' + e^{2F_0-2H} F_0''] + [e^{2F_0-2H} F_0' (F_i' + F_j' + H' - F_0')] \\ &= e^{2F_0-2H} [2F_0'^2 - 2H' F_0' + F_0'' + F_i' F_0' + F_j' F_0' + H' F_0' - F_0'^2] \\ &= e^{2F_0-2H} [F_0'^2 - H' F_0' + F_0'' + F_0' F_i' + F_0' F_j'], \end{aligned} \quad (\text{C.2})$$

$$\begin{aligned} R_{ii} &= \partial_\mu \Gamma_{ii}^\mu - \cancel{\partial_i \Gamma_{\mu i}^\mu} + \Gamma_{\mu\lambda}^\mu \Gamma_{ii}^\lambda - \Gamma_{i\lambda}^\mu \Gamma_{\mu i}^\lambda \\ &= \partial_\rho \Gamma_{ii}^\rho + \cancel{\Gamma_{i\rho}^i \Gamma_{ii}^\rho} + \Gamma_{j\rho}^j \Gamma_{ii}^\rho + \Gamma_{\rho\rho}^\rho \Gamma_{ii}^\rho + \Gamma_{t\rho}^t \Gamma_{ii}^\rho - \cancel{\Gamma_{ii}^\rho \Gamma_{\rho i}^i} - \Gamma_{i\rho}^i \Gamma_{ii}^\rho \\ &= [-e^{2F_i-2H} F_i' (2F_i' - 2H') - e^{2F_i-2H} F_i''] + [-e^{2F_i-2H} F_i' (F_j' + H' + F_0' - F_i')] \\ &= -e^{2F_i-2H} [F_i'^2 - H' F_i' + F_i'' + F_i' F_j' + F_i' F_0'], \end{aligned} \quad (\text{C.3})$$

$$\begin{aligned} R_{jj} &= \partial_\mu \Gamma_{jj}^\mu - \cancel{\partial_j \Gamma_{\mu j}^\mu} + \Gamma_{\mu\lambda}^\mu \Gamma_{jj}^\lambda - \Gamma_{j\lambda}^\mu \Gamma_{\mu j}^\lambda \\ &= \partial_\rho \Gamma_{jj}^\rho + \Gamma_{i\rho}^i \Gamma_{jj}^\rho + \cancel{\Gamma_{j\rho}^j \Gamma_{jj}^\rho} + \Gamma_{\rho\rho}^\rho \Gamma_{jj}^\rho + \Gamma_{t\rho}^t \Gamma_{jj}^\rho - \cancel{\Gamma_{jj}^\rho \Gamma_{\rho j}^j} - \Gamma_{j\rho}^j \Gamma_{jj}^\rho \\ &= [-e^{2F_j-2H} F_j' (2F_j' - 2H') - e^{2G-2H} G''] + [-e^{2F_j-2H} F_j' (F_i' + H' + F_0' - F_j')] \\ &= -e^{2F_j-2H} [F_j'^2 - H' F_j' + F_j'' + F_j' F_i' + F_j' F_0'], \end{aligned} \quad (\text{C.4})$$

$$\begin{aligned}
R_{\rho\rho} &= \partial_\mu \Gamma_{\rho\rho}^\mu - \partial_\rho \Gamma_{\mu\rho}^\mu + \Gamma_{\mu\lambda}^\mu \Gamma_{\rho\rho}^\lambda - \Gamma_{\rho\lambda}^\mu \Gamma_{\mu\rho}^\lambda \\
&= \cancel{\partial_\rho \Gamma_{\rho\rho}^\rho} - \partial_\rho \Gamma_{t\rho}^t - \partial_\rho \Gamma_{i\rho}^i - \partial_\rho \Gamma_{j\rho}^j - \cancel{\partial_\rho \Gamma_{\rho\rho}^\rho} + \Gamma_{i\rho}^i \Gamma_{\rho\rho}^\rho + \Gamma_{j\rho}^j \Gamma_{\rho\rho}^\rho + \cancel{\Gamma_{\rho\rho}^\rho \Gamma_{\rho\rho}^\rho} + \Gamma_{t\rho}^t \Gamma_{\rho\rho}^\rho \\
&\quad - \Gamma_{\rho i}^i \Gamma_{i\rho}^i - \Gamma_{\rho j}^j \Gamma_{j\rho}^j - \cancel{\Gamma_{\rho\rho}^\rho \Gamma_{\rho\rho}^\rho} - \Gamma_{\rho t}^t \Gamma_{t\rho}^t \\
&= -F_0'' - F_i'' - F_j'' + H'(F_i' + F_j' + F_0') - F_i'^2 - F_j'^2 - F_0'^2. \tag{C.5}
\end{aligned}$$

By observation, there is a clear pattern in which the i and j -components of the metric contribute to the Ricci tensor components. Therefore, we can easily generalize this to D -dimensions and obtain

$$R_{tt} = e^{2F_0 - 2H} \left(F_0'' - H'F_0' + F_0' \sum_{i=0}^n F_i' \right), \tag{C.6a}$$

$$R_{ii} = -e^{2F_i - 2H} \left(F_i'' - H'F_i' + F_i' \sum_{j=0}^n F_j' \right), \tag{C.6b}$$

$$R_{\rho\rho} = \sum_{i=0}^n (H'F_i' - F_i'' - F_i'^2). \tag{C.6c}$$

Appendix D

Verification of Ren's and Lim's Metric Solutions

D.1 Ren's Solution

Ren's metric is given as

$$ds_{\text{Ren}}^2 = \frac{l^2}{r^2} \left(-f^{p_t} dt^2 + \frac{dr^2}{f} + \sum_{i=1}^n f^{p_i} dx_i^2 \right), \quad f = 1 - \left(\frac{r}{r_0} \right)^{n+1}, \quad (\text{D.1})$$

where the Kasner exponents ($p_t := p_0$) satisfy

$$\sum_{i=0}^n p_i = 1, \quad (\text{D.2})$$

$$\sum_{i=0}^n p_i^2 = 1. \quad (\text{D.3})$$

We first perform the following coordinate transformation for ease of calculations:

$$\begin{aligned} \rho &= -\ln r, \\ \Rightarrow d\rho^2 &= \left(-\frac{1}{r} dr \right)^2 = \frac{1}{r^2} dr^2, \\ \Rightarrow \frac{1}{r^2} &= e^{2\rho}. \end{aligned} \quad (\text{D.4})$$

Thus the metric after transformation is

$$ds_{\text{Ren}}^2 = -l^2 e^{2\rho} f^{p_t} dt^2 + l^2 d\rho^2 + l^2 \sum_{i=1}^n e^{2\rho} f^{p_i} dx_i^2. \quad (\text{D.5})$$

Comparing the metric components to our metric ansatz, we see that

$$\begin{aligned} e^{2F_0} &= l^2 e^{2\rho} f^{p_t} \\ \Rightarrow F_0 &= \rho + \frac{p_t}{2} \ln f + \ln l, \end{aligned} \quad (\text{D.6a})$$

$$\begin{aligned} e^{2H} &= \frac{l^2}{f} \\ \Rightarrow H &= -\frac{1}{2} \ln f + \ln l, \end{aligned} \quad (\text{D.6b})$$

$$\begin{aligned} e^{2F_i} &= l^2 e^{2\rho} f^{p_i} \\ \Rightarrow F_i &= \rho + \frac{p_i}{2} \ln f + \ln l. \end{aligned} \quad (\text{D.6c})$$

Before proceeding, we remind the reader that all derivatives are with respect to the ρ -coordinate. Thus, we also have to express the function f in the new coordinates which gives us

$$\begin{aligned} f(\rho) &= 1 - \left(\frac{e^{-\rho}}{e^{-\rho_0}} \right)^{n+1} \\ &= 1 - e^{(\rho_0 - \rho)(n+1)}, \end{aligned} \quad (\text{D.7})$$

$$\begin{aligned} f'(\rho) &= \frac{df}{d\rho} \\ &= (n+1) e^{(\rho_0 - \rho)(n+1)}, \end{aligned} \quad (\text{D.8})$$

$$\begin{aligned} f''(\rho) &= \frac{d^2 f}{d\rho^2} \\ &= -(n+1)^2 e^{(\rho_0 - \rho)(n+1)}. \end{aligned} \quad (\text{D.9})$$

Plugging everything into the Ricci tensor component formulas from Eq. (2.14), we obtain:

$$\begin{aligned} R_{tt} &= e^{2\rho} f^{p_t+1} \left[\frac{f'}{2f} \left(1 + \frac{p_t}{2} \frac{f'}{f} \right) + \frac{p_t}{2} \frac{f'' f - f'^2}{f^2} + \left(1 + \frac{p_t}{2} \frac{f'}{f} \right) \sum_{i=0}^n \left(1 + \frac{p_i}{2} \frac{f'}{f} \right) \right] \\ &= e^{2\rho} f^{p_t+1} \left[\frac{f'}{2f} - \frac{p_t}{4} \frac{f'^2}{f^2} + \frac{p_t}{2} \frac{f''}{f} + \left(1 + \frac{p_t}{2} \frac{f'}{f} \right) \left(n+1 + \frac{1}{2} \frac{f'}{f} \right) \right] \\ &= e^{2\rho} f^{p_t+1} \left[n+1 + \frac{f'}{f} + \frac{p_t}{2} \frac{f''}{f} + \frac{p_t}{2} \frac{f'}{f} (n+1) \right] \\ &= e^{2\rho} f^{p_t} \left[(n+1) \left(f + \frac{p_t}{2} f' \right) + f' + \frac{p_t}{2} f'' \right] \end{aligned}$$

$$\begin{aligned}
& \left\{ \begin{array}{l} \text{Using Mathematica to simplify} \\ \downarrow \end{array} \right. \\
& = -\frac{n+1}{l^2} [-l^2 e^{2\rho} f^{p_i}], \tag{D.10}
\end{aligned}$$

$$\begin{aligned}
R_{ii} &= -e^{2\rho} f^{p_i+1} \left[\frac{f'}{2f} \left(1 + \frac{p_i f'}{2f} \right) + \frac{p_i f'' f - f'^2}{2f^2} + \left(1 + \frac{p_i f'}{2f} \right) \sum_{j=0}^n \left(1 + \frac{p_j f'}{2f} \right) \right] \\
&= -e^{2\rho} f^{p_i+1} \left[\frac{f'}{2f} - \frac{p_i f'^2}{4f^2} + \frac{p_i f''}{2f} + \left(1 + \frac{p_i f'}{2f} \right) \left((n+1) + \frac{1}{2} \frac{f'}{f} \right) \right] \\
&= -e^{2\rho} f^{p_i} \left[\frac{f'}{2} - \frac{p_i f'^2}{4f} + \frac{p_i f''}{2} + \left(f + \frac{p_i f'}{2} \right) \left((n+1) + \frac{1}{2} \frac{f'}{f} \right) \right] \\
&= -e^{2\rho} f^{p_i} \left[f' + \frac{p_i}{2} f'' + \left(f + \frac{p_i}{2} f' \right) (n+1) \right] \\
& \left\{ \begin{array}{l} \text{Using Mathematica to simplify} \\ \downarrow \end{array} \right. \\
& = -\frac{n+1}{l^2} [l^2 e^{2\rho} f^{p_i}], \tag{D.11}
\end{aligned}$$

$$\begin{aligned}
R_{\rho\rho} &= \sum_{i=0}^n \left[-\frac{f'}{2f} \left(1 + \frac{p_i f'}{2f} \right) - \frac{p_i f'' f - f'^2}{2f^2} - \left(1 + \frac{p_i f'}{2f} \right)^2 \right] \\
&= \sum_{i=0}^n \left[-\frac{f'}{2f} - \frac{p_i f'^2}{4f^2} - \frac{p_i f'' f - f'^2}{2f^2} - 1 - p_i \frac{f'}{f} - \frac{p_i^2 f'^2}{4f^2} \right] \\
&= -(n+1) \left[\frac{f'}{2f} + 1 \right] - \frac{1}{4} \frac{f'^2}{f^2} - \frac{1}{2} \frac{f'' f - f'^2}{f^2} - \frac{f'}{f} - \frac{1}{4} \frac{f'^2}{f^2} \\
&= -(n+1) \left[\frac{f'}{2f} + 1 \right] - \frac{1}{2} \frac{f''}{f} - \frac{f'}{f} \\
& \left\{ \begin{array}{l} \text{Using Mathematica to simplify} \\ \downarrow \end{array} \right. \\
& = -\frac{n+1}{l^2} \frac{l^2}{f}. \tag{D.12}
\end{aligned}$$

In the above calculations, the Kasner conditions are used in the intermediate steps to simplify the equations. Therefore, we have showed that Ren's metric is truly a solution to the field equation in Eq. (3.4)

$$R_{\mu\nu, \text{Ren}} = -\frac{n+1}{l^2} g_{\mu\nu}.$$

D.2 Lim's Solution

Lim's metric with the new coordinates is

$$ds_{\text{Lim}}^2 = \frac{l^2}{r^2} \left(-f^{\frac{\nu(D-2)+1}{D-1}} dt^2 + \frac{dr^2}{f} + \sum_{i=1}^n f^{\frac{1-\nu}{D-1}} dx_i^2 \right). \quad (\text{D.13})$$

Using a similar coordinate transformation as before, this metric may be rewritten as

$$ds_{\text{Lim}}^2 = -l^2 e^{2\rho} f^{\frac{\nu n+1}{n+1}} dt^2 + \frac{l^2}{f} d\rho^2 + l^2 \sum_{i=1}^n e^{2\rho} f^{\frac{1-\nu}{D-1}} dx_i^2, \quad (\text{D.14})$$

where we have let $D = n + 2$. Comparing with our metric again, we obtain

$$\begin{aligned} e^{2F_0} &= l^2 e^{2\rho} f^{\frac{\nu n+1}{n+1}} \\ \Rightarrow F_0 &= \rho + \frac{1}{2} \frac{\nu n+1}{n+1} \ln f + \ln l, \end{aligned} \quad (\text{D.15a})$$

$$\begin{aligned} e^{2H} &= \frac{l^2}{f} \\ \Rightarrow H &= -\frac{1}{2} \ln f + \ln l, \end{aligned} \quad (\text{D.15b})$$

$$\begin{aligned} e^{2F_i} &= l^2 e^{2\rho} f^{\frac{1-\nu}{n+1}} \\ \Rightarrow F_i &= \rho + \frac{1}{2} \frac{1-\nu}{n+1} \ln f + \ln l. \end{aligned} \quad (\text{D.15c})$$

Note that for Lim's metric, all $p_{i \neq 0}$ exponents are the same, while the p_t exponent is different. We therefore separate the time part from the summation terms for clarity. We shall not assume that any of Kasner's conditions are satisfied here as well. The Ricci tensor components are

$$\begin{aligned} R_{tt} &= e^{2\rho} f^{\frac{\nu n+1}{n+1}+1} \left[\left(1 + \frac{1}{2} \frac{\nu n+1}{n+1} \frac{f'}{f} \right)^2 + \frac{f'}{2f} \left(1 + \frac{1}{2} \frac{\nu n+1}{n+1} \frac{f'}{f} \right) \right. \\ &\quad \left. + \frac{1}{2} \frac{\nu n+1}{n+1} \frac{f''f - f'^2}{f^2} + \left(1 + \frac{1}{2} \frac{\nu n+1}{n+1} \frac{f'}{f} \right) \sum_{i=1}^n \left(1 + \frac{1}{2} \frac{1-\nu}{n+1} \frac{f'}{f} \right) \right] \\ &= e^{2\rho} f^{\frac{\nu n+1}{n+1}+1} \left[\left(1 + \frac{1}{2} \frac{\nu n+1}{n+1} \frac{f'}{f} \right)^2 + \frac{f'}{2f} \left(1 + \frac{1}{2} \frac{\nu n+1}{n+1} \frac{f'}{f} \right) + \frac{1}{2} \frac{\nu n+1}{n+1} \frac{f''f - f'^2}{f^2} \right. \\ &\quad \left. + n \left(1 + \frac{1}{2} \frac{\nu n+1}{n+1} \frac{f'}{f} \right) \left(1 + \frac{1}{2} \frac{1-\nu}{n+1} \frac{f'}{f} \right) \right] \end{aligned}$$

$$\begin{aligned}
&= e^{2\rho} f^{\frac{\nu n+1}{n+1}} \left[f + \frac{\nu n+1}{n+1} f' + \frac{1}{4} \left(\frac{\nu n+1}{n+1} \right)^2 \frac{f'^2}{f} + \frac{f'}{2} - \frac{1}{4} \frac{\nu n+1}{n+1} \frac{f'^2}{f} + \frac{1}{2} \frac{\nu n+1}{n+1} f'' \right. \\
&\quad \left. + n f \left(1 + \frac{1}{2} \frac{\nu n+1}{n+1} \frac{f'}{f} + \frac{1}{2} \frac{1-\nu}{n+1} \frac{f'}{f} + \frac{1}{4} \frac{(1-\nu)(\nu n+1)}{(n+1)^2} \frac{f'^2}{f^2} \right) \right] \\
&= e^{2\rho} f^{\frac{\nu n+1}{n+1}} \left[f(n+1) + f'' \left(\frac{1}{2} \frac{\nu n+1}{n+1} \right) + f' \left(\frac{\nu n+1}{n+1} + \frac{1}{2} + \frac{1}{2} \frac{n(\nu n+1)}{n+1} + \frac{1}{2} \frac{n(1-\nu)}{n+1} \right) \right. \\
&\quad \left. + \frac{f'^2}{4f} \left(n \frac{(1-\nu)(\nu n+1)}{(n+1)^2} + \left(\frac{\nu n+1}{n+1} \right)^2 - \frac{\nu n+1}{n+1} \right) \right] \\
&= e^{2\rho} f^{\frac{\nu n+1}{n+1}} \left[f(n+1) + f'' \left(\frac{1}{2} \frac{\nu n+1}{n+1} \right) + \frac{f'}{2(n+1)} (2\nu n + 2 + n + 1 + \nu n^2 + n + n - n\nu) \right. \\
&\quad \left. + \frac{f'^2}{4f(n+1)^2} \underbrace{(\nu n^2 + n - \nu n - \nu^2 n^2 + \nu^2 n^2 + 2\nu n + 1 - \nu n^2 - \nu n - n - 1)}_{=0} \right] \\
&= e^{2\rho} f^{\frac{\nu n+1}{n+1}} \left[f(n+1) + f'' \left(\frac{1}{2} \frac{\nu n+1}{n+1} \right) + \frac{f'}{2(n+1)} (\nu n^2 + \nu n + 3n + 3) \right] \\
&\quad \left\{ \begin{array}{l} \text{Using Mathematica to simplify} \\ \downarrow \end{array} \right. \\
&= \frac{n+1}{l^2} \left(l^2 e^{2\rho} f^{\frac{\nu n+1}{n+1}} \right) \\
&= -\frac{n+1}{l^2} g_{tt}, \tag{D.16}
\end{aligned}$$

$$\begin{aligned}
R_{\rho\rho} &= \sum_{i=1}^n \left[-\frac{f'}{2f} \left(1 + \frac{1}{2} \frac{1-\nu}{n+1} \frac{f'}{f} \right) - \frac{1}{2} \frac{1-\nu}{n+1} \frac{f'' f - f'^2}{f^2} - \left(1 + \frac{1}{2} \frac{1-\nu}{n+1} \frac{f'}{f} \right)^2 \right] \\
&\quad - \frac{1}{2} \frac{\nu n+1}{n+1} \frac{f'' f - f'^2}{f^2} - \frac{f'}{2f} \left(1 + \frac{1}{2} \frac{\nu n+1}{n+1} \frac{f'}{f} \right) - \left(1 + \frac{1}{2} \frac{\nu n+1}{n+1} \frac{f'}{f} \right)^2 \\
&= -\frac{n f'}{2f} + \frac{n}{4} \frac{1-\nu}{n+1} \frac{f'^2}{f^2} - \frac{n}{2} \frac{1-\nu}{n+1} \frac{f''}{f} - n - \frac{n}{4} \left(\frac{1-\nu}{n+1} \right)^2 \frac{f'^2}{f^2} - n \frac{1-\nu}{n+1} \frac{f'}{f} \\
&\quad - \frac{1}{2} \frac{\nu n+1}{n+1} \frac{f''}{f} - \frac{f'}{2f} + \frac{1}{4} \frac{\nu n+1}{n+1} \frac{f'^2}{f^2} - 1 - \frac{1}{4} \left(\frac{\nu n+1}{n+1} \right)^2 \frac{f'^2}{f^2} - \frac{\nu n+1}{n+1} \frac{f'}{f} \\
&= -(n+1) - \frac{f'}{f} \left(\frac{n+1}{2} + \frac{\nu n+1}{n+1} + n \frac{1-\nu}{n+1} \right) - \frac{f''}{f} \left(\frac{n}{2} \frac{1-\nu}{n+1} + \frac{1}{2} \frac{\nu n+1}{n+1} \right) \\
&\quad + \frac{f'^2}{4f^2(n+1)} \left(n(1-\nu) - n \frac{1+\nu^2-2\nu}{n+1} - \frac{\nu^2 n^2 + 1 + 2\nu n}{n+1} + \nu n + 1 \right)
\end{aligned}$$

$$\begin{aligned}
&= -(n+1) - \frac{f'}{f} \left(\frac{3+n}{2} \right) - \frac{f''}{f} \left(\frac{1}{2} \right) + \underbrace{\frac{f'^2}{4f^2(n+1)}}_{=\nabla_\rho\varphi\nabla_\rho\varphi} (n(1-\nu^2)) \\
&\quad \left\{ \begin{array}{l} \text{Using Mathematica to simplify} \\ \downarrow \end{array} \right. \\
&= -\frac{n+1}{l^2} \frac{l^2}{f} + \nabla_\rho\varphi\nabla_\rho\varphi \\
&= -\frac{n+1}{l^2} g_{\rho\rho} + \nabla_\rho\varphi\nabla_\rho\varphi, \tag{D.17}
\end{aligned}$$

$$\begin{aligned}
R_{ii} &= -e^{2\rho} f^{\frac{1-\nu}{n+1}+1} \left[\left(1 + \frac{1}{2} \frac{\nu n + 1}{n+1} \frac{f'}{f} \right) \left(1 + \frac{1}{2} \frac{1-\nu}{n+1} \frac{f'}{f} \right) + \frac{fL'}{2f} \left(1 + \frac{1}{2} \frac{1-\nu}{n+1} \frac{f'}{f} \right) \right. \\
&\quad \left. + \frac{1}{2} \frac{1-\nu}{n+1} \frac{f''f - f'^2}{f^2} + \left(1 + \frac{1}{2} \frac{1-\nu}{n+1} \frac{f'}{f} \right) \sum_{j=1}^n \left(1 + \frac{1}{2} \frac{1-\nu}{n+1} \frac{f'}{f} \right) \right] \\
&= -e^{2\rho} f^{\frac{1-\nu}{n+1}} \left[f \left(1 + \frac{1}{2} \frac{\nu n + 2 - \nu}{n+1} \frac{f'}{f} + \frac{1}{4} \frac{(1-\nu)(\nu n + 1)}{(n+1)^2} \frac{f'^2}{f^2} \right) + \frac{f'}{2} - \frac{1}{4} \frac{1-\nu}{n+1} \frac{f'^2}{f} + \frac{1}{2} \frac{1-\nu}{n+1} f'' \right. \\
&\quad \left. + nf \left(1 + \frac{1}{4} \left(\frac{1-\nu}{n+1} \right)^2 \frac{f'^2}{f^2} + \frac{1-\nu}{n+1} \frac{f'}{f} \right) \right] \\
&= -e^{2\rho} f^{\frac{1-\nu}{n+1}} \left[f' \left(\frac{1}{2} \frac{\nu n + 2 - \nu}{n+1} + \frac{1}{2} + \frac{n(1-\nu)}{n+1} \right) + \frac{1}{2} \frac{1-\nu}{n+1} f'' + f(n+1) \right. \\
&\quad \left. + \frac{f'^2}{4f} \underbrace{\left(\frac{n(1+\nu^2 - 2\nu)}{(n+1)^2} - \frac{1-\nu}{n+1} + \frac{(1-\nu)(\nu n + 1)}{(n+1)^2} \right)}_{=0} \right] \tag{D.18} \\
&= -e^{2\rho} f^{\frac{1-\nu}{n+1}} \left[\frac{f'}{2} (3-\nu) + \frac{1}{2} \frac{1-\nu}{n+1} f'' + f(n+1) \right]
\end{aligned}$$

$$\begin{aligned}
&\quad \left\{ \begin{array}{l} \text{Using Mathematica to simplify} \\ \downarrow \end{array} \right. \\
&= -\frac{n+1}{l^2} (l^2 e^{2\rho} f). \tag{D.19}
\end{aligned}$$

As seen above, Lim's metric is indeed a solution to the field equation in Eq. (3.5)

$$R_{\mu\nu, \text{Lim}} = -\frac{n+1}{l^2} g_{\mu\nu} + \nabla_\mu\varphi\nabla_\nu\varphi.$$

Appendix E

Rewriting the Field Equations

Given the assumptions for the EM and scalar fields:

$$\varphi = \varphi(\rho), \quad (\text{E.1a})$$

$$A_\mu = (A_0(\rho), 0), \quad (\text{E.1b})$$

the only non-zero components of the Maxwell tensor are (dropping the subscript "0" notation)

$$F_{\rho t} = -F_{t\rho} = A', \quad (\text{E.2})$$

$$\begin{aligned} \Rightarrow F^2 &= F_{\mu\nu} F^{\mu\nu} \\ &= g^{\mu\mu} g^{\nu\nu} F_{\mu\nu} F_{\mu\nu} \\ &= g^{tt} g^{\rho\rho} F_{t\rho} F_{t\rho} + g^{\rho\rho} g^{tt} F_{\rho t} F_{\rho t} \\ &= -2e^{-2F_0 - 2H} A'^2. \end{aligned} \quad (\text{E.3})$$

We also compute the following quantities

$$\begin{aligned} F_{i\beta} F_i^\beta &= 0, & F_{t\beta} F_t^\beta &= g^{\rho\rho} F_{t\rho} F_{t\rho} \\ & & &= e^{-2H} A'^2, & F_{\rho\beta} F_\rho^\beta &= g^{tt} F_{\rho t} F_{\rho t} \\ & & & & &= -e^{-2F_0} A'^2. \end{aligned} \quad (\text{E.4})$$

E.1 EM Field Equation

We begin with the modified Maxwell field equation in Eq. (2.8c) and recall the identity for the divergence of an antisymmetric tensor of rank (2, 0) [31] given by

$$\nabla_k A^{ik} = \frac{1}{\sqrt{|g|}} \partial_k \left(A^{ik} \sqrt{|g|} \right). \quad (\text{E.5})$$

Since the tensor product of a scalar and a rank (2, 0) tensor is still a rank (2, 0) tensor, Eq. (2.8c) simply becomes

$$\begin{aligned} \frac{1}{\sqrt{|g|}} \partial_\rho \left(e^{-2\alpha\varphi} F^{t\rho} \sqrt{|g|} \right) &= 0 \\ \Rightarrow e^{-2\alpha\varphi} F^{t\rho} \sqrt{|g|} &= \underbrace{q}_{\text{constant}}. \end{aligned} \quad (\text{E.6})$$

Rearranging gives us

$$\begin{aligned} g^{\rho\rho} g^{tt} A' &= -q e^{2\alpha\varphi} e^{-H - \sum_{i=0}^n F_i} \\ A' &= q e^{2\alpha\varphi} e^{H + F_0 - \sum_{i=1}^n F_i}, \end{aligned} \quad (\text{E.7})$$

Where q physically represents some electric charge parameter.

E.2 Scalar Field Equation

We now move on to the scalar field equation. The first term in Eq. (2.8b) may be rewritten as

$$\begin{aligned} \nabla^2 \varphi &= \nabla^\mu \nabla_\mu \varphi \\ &= g^{\mu\nu} \nabla_\nu (\partial_\mu \varphi) \\ &= g^{\mu\nu} [\partial_\nu \partial_\mu \varphi - \Gamma_{\nu\mu}^\gamma \partial_\gamma \varphi] \\ &= g^{\rho\rho} [\partial_\rho \partial_\rho \varphi - \Gamma_{\rho\rho}^\gamma \partial_\gamma \varphi] + g^{tt} [-\Gamma_{tt}^\gamma \partial_\gamma \varphi] + \sum_{i=1}^n g^{ii} [-\Gamma_{ii}^\gamma \partial_\gamma \varphi] \\ &= e^{-2H} (\varphi'' - H' \varphi') - e^{-2F_0} (-e^{2F_0 - 2H} F_0' \varphi') + \sum_{i=1}^n e^{-2F_i} (e^{2F_i - 2H} F_i' \varphi') \end{aligned}$$

$$= e^{-2H} \left[\varphi'' + \left(\sum_{i=0}^n F'_i - H' \right) \varphi' \right]. \quad (\text{E.8})$$

Therefore, substituting back into the scalar field equation, Eq. (2.8b), and rearranging gives

$$\begin{aligned} \varphi'' + \left(\sum_{i=0}^n F'_i - H' \right) \varphi' &= 2\alpha e^{-2\alpha\varphi - 2F_0} A'^2 + 2\Lambda\beta e^{2\beta\varphi + 2H} \\ &= 2\alpha e^{-2\alpha\varphi - 2F_0} (q^2 e^{4\alpha\varphi} e^{2H + 2F_0 - 2\sum_{i=1}^n F_i}) + 2\Lambda\beta e^{2\beta\varphi + 2H} \\ &= 2\alpha q^2 e^{2\alpha\varphi + 2H - 2\sum_{i=1}^n F_i} + 2\Lambda\beta e^{2\beta\varphi + 2H}. \end{aligned} \quad (\text{E.9})$$

E.3 Einstein Field Equations

For the Einstein field equation in Eq. (2.8a), we simply substitute the obtained Ricci tensor components for our metric in Eq. (2.14) and rearrange accordingly.

For R_{tt} , we get

$$\begin{aligned} e^{2F_0 - 2H} \left[F_0'' - H' F_0' + F_0' \sum_{i=0}^n F'_i \right] &= -\frac{2\Lambda}{D-2} e^{2\beta\varphi + 2F_0} + 2e^{-2\alpha\varphi} e^{-2H} A'^2 \left[\frac{D-3}{D-2} \right] \\ \Rightarrow F_0'' &= F_0' \left(H' - \sum_{i=0}^n F'_i \right) - \frac{2\Lambda}{D-2} e^{2\beta\varphi + 2H} + 2q^2 e^{2\alpha\varphi + 2H - 2\sum_{i=1}^n F_i} \left[\frac{D-3}{D-2} \right]. \end{aligned} \quad (\text{E.10})$$

For $R_{\rho\rho}$, we get

$$\begin{aligned} \sum_{i=0}^n (H' F'_i - F_i'' - F_i'^2) &= \frac{2\Lambda}{D-2} e^{2\beta\varphi + 2H} - 2e^{-2\alpha\varphi} e^{-2F_0} A'^2 \left[\frac{D-3}{D-2} \right] + \nabla_\rho \varphi \nabla_\rho \varphi \\ \Rightarrow \sum_{i=0}^n (H' F'_i - F_i'' - F_i'^2) &= \frac{2\Lambda}{D-2} e^{2\beta\varphi + 2H} - 2q^2 e^{2\alpha\varphi + 2H - 2\sum_{i=1}^n F_i} \left[\frac{D-3}{D-2} \right] + \varphi'^2. \end{aligned} \quad (\text{E.11})$$

For R_{ii} , where $i \neq 0$, we get

$$-e^{2F_i - 2H} \left[F_i'' - H' F'_i + F_i' \sum_{j=0}^n F'_j \right] = \frac{2\Lambda}{D-2} e^{2\beta\varphi + 2F_i} + \frac{2}{D-2} e^{-2\alpha\varphi} e^{2F_i - 2F_0 - 2H} A'^2$$

$$\Rightarrow F''_{i \neq 0} = F'_{i \neq 0} \left(H' - \sum_{j=0}^n F'_j \right) - \frac{2\Lambda}{D-2} e^{2\beta\varphi+2H} - \frac{2q^2}{D-2} e^{2\alpha\varphi+2H-2\sum_{i=1}^n F_i}. \quad (\text{E.12})$$

It may seem that one more equation may be obtained by equating the Ricci scalar in Eq. (2.15) and the trace of the field equation in Eq. (2.6):

$$\begin{aligned} e^{-2H} \sum_{i=0}^n \left(2H' F'_i - 2F''_i - F_i'^2 - F'_i \sum_{j=0}^n F'_j \right) &= \\ &= \frac{2D}{D-2} \Lambda e^{2\beta\varphi} + e^{-2H} \varphi'^2 - 2 \frac{D-4}{D-2} e^{-2\alpha\varphi} e^{-2F_0-2H} A'^2 \\ \Rightarrow \sum_{i=0}^n \left(2H' F'_i - 2F''_i - F_i'^2 - F'_i \sum_{j=0}^n F'_j \right) &= \\ &= \frac{2D}{D-2} \Lambda e^{2\beta\varphi+2H} + \varphi'^2 - 2q^2 \frac{D-4}{D-2} e^{2\alpha\varphi+2H-2\sum_{i=1}^n F_i}. \end{aligned} \quad (\text{E.13})$$

However, if we take Eq. (E.11) from the $R_{\rho\rho}$ component and subtract it from Eq. (E.13) above, this reduces to

$$\sum_{i=0}^n \left(H' F'_i - F''_i - F'_i \sum_{j=0}^n F'_j \right) = \frac{2\Lambda(D-1)}{D-2} e^{2\beta\varphi+2H} + \frac{2q^2}{D-2} e^{2\alpha\varphi+2H-2\sum_{i=1}^n F_i}. \quad (\text{E.14})$$

But this does not give us any new information because a closer look shows that

$$\text{Eq. (E.14)} = \text{Eq. (E.10)} + \sum_{i=1}^n \text{Eq. (E.12)}. \quad (\text{E.15})$$

In summary, we have the following $(4+n)$ equations which fully describes Einstein-Maxwell-Dilaton gravity under our ansatz (n for the $F''_{i \neq 0}$: components)

$$\begin{aligned} A' &= q e^{2\alpha\varphi} e^{H+F_0-\sum_{i=1}^n F_i}, \\ \varphi'' &= \left(H' - \sum_{i=0}^n F'_i \right) \varphi' + 2\alpha q^2 e^{2\alpha\varphi+2H-2\sum_{i=1}^n F_i} + 2\Lambda\beta e^{2\beta\varphi+2H}, \\ F''_0 &= F'_0 \left(H' - \sum_{i=0}^n F'_i \right) - \frac{2\Lambda}{D-2} e^{2\beta\varphi+2H} + 2q^2 \left(\frac{D-3}{D-2} \right) e^{2\alpha\varphi+2H-2\sum_{i=1}^n F_i}, \\ F''_{i \neq 0} &= F'_{i \neq 0} \left(H' - \sum_{j=0}^n F'_j \right) - \frac{2\Lambda}{D-2} e^{2\beta\varphi+2H} - \frac{2q^2}{D-2} e^{2\alpha\varphi+2H-2\sum_{i=1}^n F_i}, \end{aligned}$$

$$\sum_{i=0}^n (H' F_i' - F_i'' - F_i'^2) = \frac{2\Lambda}{D-2} e^{2\beta\varphi+2H} - 2q^2 \left(\frac{D-3}{D-2} \right) e^{2\alpha\varphi+2H-2\sum_{i=1}^n F_i} + \varphi'^2.$$

The first two equations correspond to the EM and scalar field respectively, and the other equations come from the Einstein field equation.

Appendix F

Solution to the Liouville Differential Equation

The Liouville differential equation

$$\phi''(\rho) = ke^{2\phi(\rho)}, \quad (\text{F.1})$$

where k is a constant, has the general solution

$$\phi(\rho) = -b(\rho + \rho_0) - \ln \left(\frac{k}{4b^2c} - ce^{-2b(\rho+\rho_0)} \right), \quad (\text{F.2})$$

where b and c are the integration constants and ρ_0 is the “zero” of the ρ -variable.

F.1 For Positive k

In this case, we may simplify by shifting the zero of ρ in the following manner:

$$\begin{aligned} \frac{k}{4b^2c} &= \frac{c}{e^{2b\rho_0}} \\ \Rightarrow \frac{c}{e^{b\rho_0}} &= \frac{\sqrt{k}}{2b}. \end{aligned} \quad (\text{F.3})$$

Therefore we have

$$\ln \left(\frac{k}{4b^2c} - ce^{-2b(\rho+\rho_0)} \right) = \ln (ce^{-2b\rho_0} - ce^{-2b(\rho+\rho_0)})$$

$$\begin{aligned}
&= \ln [ce^{-b(\rho+\rho_0)} (e^{-b(\rho_0-\rho)} - e^{-b(\rho_0+\rho)})] \\
&= -b(\rho + \rho_0) + \ln \left[\frac{\sqrt{k}}{2b} (e^{b\rho} - e^{-b\rho}) \right] \\
&= -b(\rho + \rho_0) + \ln \left(\frac{\sqrt{k}}{b} \sinh(b\rho) \right). \tag{F.4}
\end{aligned}$$

Substituting back into Eq. (F.2), the first term cancels and we get

$$\phi_+ = -\ln \left(\frac{\sqrt{k}}{b} \sinh(b\rho) \right). \tag{F.5}$$

F.2 For Negative k

For the negative case, we may simplify by shifting the zero of ρ in the following manner instead:

$$\begin{aligned}
\frac{-k}{4b^2c} &= \frac{c}{e^{2b\rho_0}} \\
\Rightarrow \frac{c}{e^{b\rho_0}} &= \frac{\sqrt{-k}}{2b}. \tag{F.6}
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\ln \left(-\frac{-k}{4b^2c} - ce^{-2b(\rho+\rho_0)} \right) &= \ln (-ce^{-2b\rho_0} - ce^{-2b(\rho+\rho_0)}) \\
&= \ln [-ce^{-b(\rho+\rho_0)} (e^{-b(\rho_0-\rho)} + e^{-b(\rho_0+\rho)})] \\
&= -b(\rho + \rho_0) + \ln \left[-\frac{\sqrt{-k}}{2b} (e^{b\rho} + e^{-b\rho}) \right] \\
&= -b(\rho + \rho_0) + \ln \left(\frac{\sqrt{-k}}{b} \cosh(b\rho) \right), \tag{F.7}
\end{aligned}$$

where we have chosen to take the negative value of the squareroot to cancel out the minus sign. Substituting back into Eq. (F.2), the first term cancels again and we get

$$\phi_- = -\ln \left(\frac{\sqrt{-k}}{b} \cosh(b\rho) \right). \tag{F.8}$$

Appendix G

Solutions for Cases 2, 3 and 4

G.1 Case 2: $\Lambda = 0$

Setting $\Lambda = 0$, Eq. (4.13) is simplified to

$$\phi_1'' = -2q^2 \left(\frac{1}{D-2} - \alpha\beta \right) e^{2\phi_2}, \quad (\text{G.1a})$$

$$\phi_2'' = 2q^2 \left(\frac{D-3}{D-2} + \alpha^2 \right) e^{2\phi_2}, \quad (\text{G.1b})$$

$$\phi_3'' = -2q^2 (1 - \alpha^2) e^{2\phi_2}. \quad (\text{G.1c})$$

According to Eq. (4.15a), the solutions are

$$\phi_1 = \frac{(\alpha\beta - \frac{1}{D-2})}{(\frac{D-3}{D-2} + \alpha^2)} \phi_2 + c_2\rho + c_3, \quad (\text{G.2a})$$

$$\phi_2 = -\ln \left(\sqrt{\frac{2q^2}{b^2} \left(\frac{D-3}{D-2} + \alpha^2 \right) \sinh(b\rho)} \right), \quad (\text{G.2b})$$

$$\phi_3 = \frac{(\alpha^2 - 1)}{(\frac{D-3}{D-2} + \alpha^2)} \phi_2 + d_2\rho + d_3, \quad (\text{G.2c})$$

where c_2 , c_3 , d_2 and d_3 are arbitrary integration constants. Rearranging the change of variables in Eq. (4.12), we reobtain

$$\varphi = \frac{1}{2\alpha - \beta} \left[\phi_2 + \frac{(\alpha^2 - 1)}{(\frac{D-3}{D-2} + \alpha^2)} \phi_2 - \frac{(\alpha\beta - \frac{1}{D-2})}{(\frac{D-3}{D-2} + \alpha^2)} \phi_2 + (d_2 - c_2)\rho + d_3 - c_3 \right]$$

$$= \frac{\alpha}{\left(\frac{D-3}{D-2} + \alpha^2\right)} \phi_2 + \frac{(d_2 - c_2)\rho + d_3 - c_3}{2\alpha - \beta}, \quad (\text{G.3a})$$

$$\begin{aligned} F_{i \neq 0} &= \frac{1}{D-2} \left[\frac{(\alpha^2 - 1)}{\left(\frac{D-3}{D-2} + \alpha^2\right)} \phi_2 - \frac{\alpha^2}{\left(\frac{D-3}{D-2} + \alpha^2\right)} \phi_2 + d_2\rho + d_3 - \frac{\alpha}{2\alpha - \beta} \left((d_2 - c_2)\rho + d_3 - c_3 \right) \right] \\ &= \frac{1}{D-2} \left[-\frac{1}{\left(\frac{D-3}{D-2} + \alpha^2\right)} \phi_2 + d_2\rho + d_3 - \frac{\alpha}{2\alpha - \beta} \left((d_2 - c_2)\rho + d_3 - c_3 \right) \right], \end{aligned} \quad (\text{G.3b})$$

$$\begin{aligned} F_0 &= \phi_2 - \frac{\alpha^2}{\left(\frac{D-3}{D-2} + \alpha^2\right)} \phi_2 - \frac{\alpha}{2\alpha - \beta} \left((d_2 - c_2)\rho + d_3 - c_3 \right) \\ &= \frac{D-3}{D-2} \frac{1}{\left(\frac{D-3}{D-2} + \alpha^2\right)} \phi_2 - \frac{\alpha}{2\alpha - \beta} \left((d_2 - c_2)\rho + d_3 - c_3 \right). \end{aligned} \quad (\text{G.3c})$$

Substituting into Eq. (4.11e), we get the constraint equation for the arbitrary constants:

$$\begin{aligned} &c^2 (\alpha^2(3D - 7) - D + 2) + 2cd (-3\alpha^2 + 5\alpha\beta + \alpha^2D - 2\alpha\beta D + D - 2) + \\ &d^2 (\alpha^2 + 2\alpha\beta - (\alpha^2 + 1)D + \beta^2(D - 3) + 2) = \frac{b^2(D-2)(D-2)(\beta-2\alpha)^2}{\alpha^2(D-2) + D-3}. \end{aligned} \quad (\text{G.4})$$

We can set c_3 and d_3 to 0, which can also be interpreted as a rescaling of the coordinates. Dropping the subscripts on $c_2 \rightarrow c$ and $d_2 \rightarrow d$, the final metric solution is then

$$\begin{aligned} ds^2 &= -e^{-\frac{2\alpha(d-c)}{2\alpha-\beta}\rho} G(\rho)^{\frac{D-3}{(D-3+(D-2)\alpha^2)}} dt^2 \\ &\quad + e^{-2d\rho} G(\rho)^{-\frac{1}{(D-3+(D-2)\alpha^2)}} d\rho^2 \\ &\quad + \sum_{i=1}^n e^{-\frac{2}{D-2}\left(\frac{\alpha(d+c)-\beta d}{2\alpha-\beta}\right)\rho} G(\rho)^{-\frac{1}{(D-3+(D-2)\alpha^2)}} dx_i^2, \end{aligned} \quad (\text{G.5})$$

with $G(\rho) = \frac{2q^2}{b^2} \left(\frac{D-3}{D-2} + \alpha^2\right) \sinh^2(b\rho)$. The corresponding scalar field is

$$\varphi = -\frac{\alpha}{\left(\frac{D-3}{D-2} + \alpha^2\right)} \ln \left(\frac{1}{b} \sqrt{2q^2 \left(\frac{D-3}{D-2} + \alpha^2\right) \sinh^2(b\rho)} \right) + \frac{(d-c)\rho}{2\alpha - \beta}, \quad (\text{G.6})$$

and the electric field is

$$E = -\frac{b^2(D-2)\text{csch}^2(b\rho)}{2q(D-3 + \alpha^2(D-2))}. \quad (\text{G.7})$$

G.2 Case 3: $\alpha = \beta = 1$, $D = 3$

In this case, Eq. (4.13) is simplified to

$$\phi_1'' = -2\Lambda e^{2\phi_1}, \quad (\text{G.8a})$$

$$\phi_2'' = 2q^2 e^{2\phi_2}, \quad (\text{G.8b})$$

$$\phi_3'' = 0. \quad (\text{G.8c})$$

For a positive Λ , according to Eq. (4.15a), the solutions are

$$\phi_1 = -\ln \left(-\frac{\sqrt{2\Lambda}}{b_1} \cosh(b_1\rho) \right), \quad (\text{G.9a})$$

$$\phi_2 = -\ln \left(\frac{q\sqrt{2}}{b_2} \sinh(b_2\rho) \right), \quad (\text{G.9b})$$

$$\phi_3 = b_3\rho + b_4, \quad (\text{G.9c})$$

where b_3 and b_4 are arbitrary integration constants. Rearranging the change of variables in Eq. (4.12), we reobtain

$$\varphi = b_3\rho + b_4 + \ln \left(-\frac{b_2\sqrt{\Lambda}}{qb_1} \frac{\cosh(b_1\rho)}{\sinh(b_2\rho)} \right), \quad (\text{G.10a})$$

$$F_i = \ln \left(-\frac{qb_1}{b_2\sqrt{\Lambda}} \frac{\sinh(b_2\rho)}{\cosh(b_1\rho)} \right), \quad (\text{G.10b})$$

$$F_0 = -b_3\rho - b_4 - \ln \left(-\frac{\sqrt{2\Lambda}}{b_1} \cosh(b_1\rho) \right). \quad (\text{G.10c})$$

Substituting into Eq. (4.11e), we get the constraint equation for the arbitrary constants:

$$b_1^2 - b_2^2 - b_3^2 = 0. \quad (\text{G.11})$$

We can set b_4 to 0, which can also be interpreted as a rescaling of the coordinates. The final metric solution is then

$$ds^2 = -e^{-2b_3\rho} \frac{1}{M(\rho)} dt^2 + e^{-2b_3\rho} \left(\frac{2q^2 \sinh^2(b_2\rho)}{b_2^2 M(\rho)} \right) d\rho^2 + \left(\frac{2q^2 \sinh^2(b_2\rho)}{b_2^2 M(\rho)} \right) dx_i^2, \quad (\text{G.12})$$

with

$$M(\rho) = \begin{cases} \frac{2\Lambda}{b_1^2} \cosh^2(b_1\rho), & \text{for } \Lambda > 0, \\ \frac{2\Lambda}{b_1^2} \sinh^2(b_1\rho), & \text{for } \Lambda < 0. \end{cases} \quad (\text{G.13})$$

The corresponding scalar field is

$$\varphi = b_3\rho + \ln \left(\frac{b_2\sqrt{M(\rho)}}{\sqrt{2q} \sinh(b_2\rho)} \right), \quad (\text{G.14})$$

and the electric field is

$$E = -\frac{b_2^2 \text{csch}^2(b_2\rho)}{2q}. \quad (\text{G.15})$$

G.3 Case 4: $\alpha\beta = 1$, $D = 3$

In this case, Eq. (4.13) is simplified to

$$\phi_1'' = 2\Lambda \left(\frac{1}{\alpha^2} - 2 \right) e^{2\phi_1}, \quad (\text{G.16a})$$

$$\phi_2'' = 2q^2 \alpha^2 e^{2\phi_2}, \quad (\text{G.16b})$$

$$\phi_3'' = -2q^2 (1 - \alpha^2) e^{2\phi_3}. \quad (\text{G.16c})$$

For $2\Lambda(2\alpha^2 - 1) > 0$, according to Eq. (4.15a), the solutions are

$$\phi_1 = -\ln \left(-\frac{\sqrt{2\Lambda(2 - \frac{1}{\alpha^2})}}{b_1} \cosh(b_1\rho) \right), \quad (\text{G.17a})$$

$$\phi_2 = -\ln \left(\frac{q\alpha\sqrt{2}}{b_2} \sinh(b_2\rho) \right), \quad (\text{G.17b})$$

$$\phi_3 = \frac{(1 - \alpha^2)}{\alpha^2} \ln \left(\frac{q\alpha\sqrt{2}}{b_2} \sinh(b_2\rho) \right) + c_2\rho + c_3, \quad (\text{G.17c})$$

where c_2 and c_3 are arbitrary integration constants. Rearranging the change of variables in Eq. (4.12), we reobtain

$$\varphi = \ln \left[\frac{\left(-\frac{\sqrt{2\Lambda(2 - \frac{1}{\alpha^2})}}{b_1} \cosh(b_1\rho) \right)^{\frac{\alpha}{2\alpha^2 - 1}}}{\left(\frac{q\alpha\sqrt{2}}{b_2} \sinh(b_2\rho) \right)^{\frac{1}{\alpha}}} \right] + \frac{\alpha(c_2\rho + c_3)}{2\alpha^2 - 1}, \quad (\text{G.18a})$$

$$F_i = \ln \left[\frac{\left(-\frac{\sqrt{2\Lambda(2-\frac{1}{\alpha^2})}}{b_1} \cosh(b_1\rho) \right)^{-\frac{\alpha^2}{2\alpha^2-1}}}{\left(\frac{q\alpha\sqrt{2}}{b_2} \sinh(b_2\rho) \right)^{-\frac{1}{\alpha^2}}} \right] + \frac{(\alpha^2-1)(c_2\rho+c_3)}{2\alpha^2-1}, \quad (\text{G.18b})$$

$$F_0 = \ln \left[\left(-\frac{\sqrt{2\Lambda(2-\frac{1}{\alpha^2})}}{b_1} \cosh(b_1\rho) \right)^{\frac{-\alpha^2}{2\alpha^2-1}} \right] - \frac{\alpha^2(c_2\rho+c_3)}{2\alpha^2-1}. \quad (\text{G.18c})$$

Substituting into Eq. (4.11e), we get the constraint equation for the arbitrary constants:

$$\alpha^4 b_1^2 + (1-2\alpha^2) b_2^2 - \alpha^4 c^2 = 0. \quad (\text{G.19})$$

We can set c_3 to 0, which can also be interpreted as a rescaling of the coordinates. Dropping the subscript on $c_2 \rightarrow c$, the final metric solution is then

$$ds^2 = -e^{-\frac{2\alpha^2(c\rho)}{2\alpha^2-1}} J(\rho)^{\frac{-\alpha^2}{2\alpha^2-1}} dt^2 + e^{\frac{2(c\rho)}{2\alpha^2-1}} \left(\frac{2q^2\alpha^2}{b_2^2} \sinh^2(b_2\rho) \right)^{\frac{1}{\alpha^2}} J(\rho)^{-\frac{2\alpha^2}{2\alpha^2-1}} d\rho^2 \\ + e^{\frac{2(\alpha^2-1)(c\rho)}{2\alpha^2-1}} \left(\frac{2q^2\alpha^2}{b_2^2} \sinh^2(b_2\rho) \right)^{\frac{1}{\alpha^2}} J(\rho)^{-\frac{\alpha^2}{2\alpha^2-1}} dx^2, \quad (\text{G.20})$$

where

$$J(\rho) = \begin{cases} \frac{2\Lambda(2\alpha^2-1)}{\alpha^2 b_1^2} \cosh^2(b_1\rho), & \text{for } 2\Lambda(2\alpha^2-1) > 0, \\ \frac{2\Lambda(1-2\alpha^2)}{\alpha^2 b_1^2} \sinh^2(b_1\rho), & \text{for } 2\Lambda(2\alpha^2-1) < 0. \end{cases} \quad (\text{G.21})$$

The corresponding scalar field is

$$\varphi = \frac{\alpha}{2(2\alpha^2-1)} \ln J(\rho) - \frac{1}{\alpha} \ln \left(\frac{q\alpha\sqrt{2}}{b_2} \sinh(b_2\rho) \right) + \frac{\alpha c\rho}{2\alpha^2-1}, \quad (\text{G.22})$$

and the electric field is

$$E = -\frac{b_2^2 \text{csch}^2(b_2\rho)}{2\alpha^2 q}. \quad (\text{G.23})$$

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