Supersymmetric Quantum Mechanics and Morse Theory: A Review

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This thesis is dedicated to
my parents
for supporting my education
so I may chase my *dreams*
without unnecessary worries.

"Life," said Marvin dolefully, "loathe it or ignore it, you can't like it." – The Hitchhiker's Guide to the Galaxy

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Abstract

In this review, we consider the supersymmetry formalism for zero and one $(0 + 1)$ dimension quantum field theory. Motivated strongly by the unification of the fundamental forces, supersymmetry is a formalism that describes a symmetry in the exchange of bosons with fermions and vice-versa. In particle physics, it provides an extension to the Poincaré[1] group of transformations to what is called the super-Poincaré algebra[2]. We will study the several characteristic features of supersymmetry that allow for obtaining analytic solutions for certain class of calculations which would have been otherwise difficult to solve in standard theories. Our discussion begins in Chapter I by providing an overview to the study of supersymmetry, followed by a discussion on the necessary mathematical requisites to understand the formalism. Specifically, we will be discussing some of the results of Morse[3] Theory to provide the necessary background for our later discussion. In Chapter II, we will begin discussion proper on supersymmetry by studying the formalism defined on a zero-dimensional base manifold. As the various mathematical results unique to supersymmetry may be rigorously defined in zero-dimensions, similar calculations in higher dimensions can be explored and justified by extension. Following which, in Chapter III, we first show that *a one-dimensional quantum field theory is equivalent to studying quantum mechanics*. Which would allow us to relate familiar concepts from quantum mechanics to the quantum field theory. We will then study supersymmetric ground states in one-dimensions quantum field theory. The study of *supersymmetric ground states* helps us understand the conditions under which supersymmetry is broken at the infrared limit given that supersymmetry does not already manifest in nature. It is found that the number of supersymmetric ground states obtained via perturbative methods are bounded from above. We then associate the ground states with the topological features of our manifold and identify a correspondence with Morse Theory. The superpotentials in the supersymmetric theory are found to be equivalent to Morse functions. This correspondence is actively discussed in literature [1–3].

[1] Jules Henri Poincaré (1854 - 1912)

[2] We say that the super-Poincaré group contains the Poincaré group of spacetime symmetries as a subgroup.

[3] Harold Calvin Marston Morse (1892 - 1977)

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Chapter I

Introduction

In this chapter, we will be providing some context to the study of supersymmetry and define what are sigma models in quantum field theories. Then, in order to ensure that the review is self-contained, we will develop the necessary mathematical tools required to understand supersymmetric quantum field theories: differential geometry, topological invariants and lastly Morse Theory. Morse Theory is the study of the topology of a manifold by observing the properties of functions defined on this abstract space. The analysis of the critical points of what are known as Morse functions can relate to us the topology of that space. Morse Theory has the advantage that critical points analysis of polynomial functions can be easily worked out and it is in agreement with other methods of analysis. We would then end off our discussion of this chapter by highlighting some of the key features in quantum field theories which will be the essential formalism in this review.

The study of supersymmetry in zero and one-dimensional quantum field theory provides for a rigorously formulated theory in which the mathematical results are extended into higher dimensions. We will begin our discussion by defining supersymmetry.

1.1 Supersymmetry

Currently, the most successful theory we have in the field of physics is the Standard Model. The Standard Model is a quantum field theory that describes all currently known fundamental constituents of matter and forces as fields. Under this formalism, nature is described in terms of fields and the particles that we observe are in fact field excitations. Furthermore, the Standard Model describes interactions between fields as gauge interactions, i.e. the force carriers (photons, $W^{\pm}$ and Z) are represented as gauge bosons[1] while matter is made up of fermions (quarks and leptons). To further elaborate, fermions are particles with half-integer spins and hence obey the Fermi-Dirac statistics, while bosons are particles with integer spins obeying the Bose-Einstein statistics. It's not apparent

[1]Gauge bosons mediate interactions between elementary matter particles described by gauge theories.

from our current understand why such a distinction between matter and force carriers exists, but it is possible that understanding supersymmetry may help provide insights on this matter.

Supersymmetry is a symmetry between forces (bosons) and matter (fermions) in which the theory is invariant when we interchange bosons and fermions. The study of supersymmetry historically began as a method to expand the Poincaré group of symmetries in an attempt to combine spacetime and internal symmetries in quantum field theories. This was motivated by the Coleman-Mandula theorem [6]: the Coleman-Mandula theorem is a no-go theorem in theoretical physics that suggests that it is impossible to expand the symmetry groups in quantum field theories, satisfying a set of assumptions, using scalar generators. Supersymmetry circumvents this problem by introducing additional symmetry generators that are spinors. As the generators of supersymmetry are fermionic, they carry half-integer angular momentum which are associated to spacetime symmetries [7, 8]. It is distinct from the spacetime symmetry that describes particle under the inhomogeneous Lorentz group (or Poincaré group) of symmetries which are bosonic while encompassing it as a subgroup.

Supersymmetry from the mathematical perspective, is a formalism that introduces non-commuting fermionic odd variables on the functional space. The additional symmetries of the odd variables introduced are hence naturally fermionic by construction. This is striking as in contrast to standard quantum field theories where there exists only commuting bosonic symmetries. The fermionic odd variables and ordinary bosonic variables can be formulated over an enlarged abstract space known as the *superspace*, i.e. they are functions defined on the superspace. The superspace is an extended coordinate space with the standard bosonic spacetime coordinates x^μ, and fermionic coordinates $\theta, \overline{\theta}$. Hence, the fields can be expressed as functions that takes in arguments on the extended coordinate space, e.g. the scalar field $\phi := \phi(x^\mu, \theta, \overline{\theta})$. While it's possible to formulate supersymmetric models starting from the superspace interpretation, that will not be the approach taken here. Instead, we will start from a given set of bosonic and fermionc fields and define the appropriate Lagrangian, then analyse its properties.

One common property of supersymmetric models is that they predict the existence of superpartners: in particle physics, a supersymmetric standard model predicts that the existing particles described by the Standard Model are coupled in boson-fermion pairs. In doing so, the number of particles in the Standard Model is doubled. This is vaguely reminiscent of the time when Dirac hypothesised the existence of anti-particles from the Dirac equation. The discovery of anti-particles then solidified the Dirac equation as a relativistic equation of motion that describes fermions. Similarly, an experimental discovery of the superpartners of the existing particles would also provide basis for supersymmetry.

However, the absence of these hypothesised boson-fermion pairs in nature leads to two possible conclusions: 1. while supersymmetry may be an elegant theory, it simply does not manifest in nature; or 2. supersymmetry is spontaneously broken in nature at low energies. The second conclusion motivates the search for the superpartners at the Large Hadron Collider (LHC) at high energies under the Minimal Supersymmetric Standard Model (MSSM).[2] We will however not be addressing the MSSM in this review. Concerning us, what the second conclusion further implies is that the ground states are not invariant under supersymmetric variations and are hence not supersymmetric ground states. Given that energy and mass are equivalent descriptions, this means that the superpartners do not have the same mass. In order to understand this in greater detail, we will be looking at the supersymmetric ground states in the $(0+1)$-dimensional theory in Chapter III. Nonetheless, the study of supersymmetry has far reaching applications in condensed matter physics and statistical physics [9–13]. In this review, we will relate the supersymmetry formalism to the study of Morse Theory, a theory in topological manifolds, following [1]. It is shown that Morse functions are realised as superpotentials in the supersymmetric theory.

In order to study the zero and one-dimensional supersymmetric quantum field theory, we need first define the objects of the theory starting from the differential geometry perspective. We first define the base manifold M with d-dimensions to be the space that parametrises a field – pertaining to this review, our discussion is limited to $d = 0, 1$. The $d = 1$ case is more specifically a $(0 + 1)$ theory for which the field is parametrised by t which we interpret to be time.

1.2 Sigma Models

Historically, the term *sigma models* (or σ models) was first introduced in the phenomenological model for β-decay by Gell-Mann and Lévy [14] in which the supposed σ meson particle (a scalar particle) was first introduced by Julian Schwinger [15].

On the manifold M, we may further define the space of maps

$$x : M \to N, \tag{1.2.1}$$

where x is a scalar field[3] that maps the base manifold M onto the target space N. In the context of string theory, the strings are defined on the base manifold (the 1+1 dimensional Riemann surface or worldsheet) and are mapped onto the Riemannian manifold (N, g), where g is the Riemannian metric on N. The integration over the space of scalar maps constitutes what is known as σ-models.

In order to carry out such an integration, we will first need to define a measure on the space of maps. The measure is typically weighted by e^{-S}, where S is a functional on

[2] Note that the LHC is currently running at 13 TeV energies at centre of momentum frame, however there no positive signs in the search for superpartners with masses predicted by the MSSM currently.

[3] For zero and one-dimensions, our discussion is trivially restricted to scalar fields.

the space of fields, known as the action. If M is a Minkowski space, the measure is the modified weight of e^{iS}.

It can be shown that we may associate M with an Euclidean space via a Wick rotation in which we take $t \to i\tau$. Hence, taking the 2 dimensional case as an example, the metric is mapped from $-dt^2 + dx^2$ to $d\tau^2 + dx^2$. The theory hence describes a Riemannian manifold instead of a pseudo-Riemannian theory that we might be familiar with in Relativity.

1.3 Mathematical Preliminaries

We briefly summarise the key ideas in differential geometry and the study of topology that will be required for the discussion in this review. The discussion here follows closely to [4].

The most general structure studied in physics are *topological spaces*. Informally, topology is the study of classifying spaces, while manifolds form a subset of topological spaces. Topological spaces are classified by identifying the classes and number topological invariants. We will assume a knowledge of what is a topology or at least a sense of what it is and move onto defining what is the study of differential geometry.

Differential geometry is the study of *differentiable manifolds*. Formally, a manifold M is a set of points in which for each point $p \in M$, p has an open neighbourhood U. The neighbourhood U_i is an open set equipped with a continuous one-to-one map f_i onto an open set of $\mathbb{R}^n$ (a bijective mapping), for some number n. n is typically the dimension of the manifold, $\dim M = n$. Or simply said, it is locally flat (Euclidean) in the neighbourhood of $p \in M$. This definition extends to include not just trivially $\mathbb{R}^n$ but also smooth curves, circles, spheres $(S^n, n \geq 1)$ and tori $(T^n, n \geq 2)$, etc.

The neighbourhood U being *homeomorphic* to $\mathbb{R}^n$ defines what is meant by a chart. Two topological spaces are homeomorphic to each other if we can deform one continuously to the other.

Typically, more than one chart is required to cover a manifold; a manifold is hence a collection of two or more charts. This is the atlas, denoted by (U_i, f_i). Hence we may also say that a manifold is a topogical space which is locally homeomorphic to $\mathbb{R}^n$.

By differentiable, we mean that the set of transition functions $f_i \circ f_j^{-1}$, between overlapping charts $U_i, U_j, U_i \cap U_j \neq \emptyset$ is a C^∞ (continuously differentiable) module. In practical terms, this condition is relaxed as we only require it to be finitely differentiable as required: a C^k module where k is a finite integer.

1.3.1 Differential Forms

Given a manifold M, let us denote the tangent vector space at $p \in M$ to be T_pM. There then exists the dual vector space, the cotangent space denoted by T_p^*M. The elements in the cotangent space are the *one-forms*: the basis one-forms being linear maps from T_pM to $\mathbb{R}$.

$$\tilde{\omega} : T_pM \to \mathbb{R}. \tag{1.3.1}$$

We may further generalise this to allow for higher ordered functions by introducing the totally anti-symmetric r-forms. A differential form of order r or an r-form is a totally anti-symmetric $(0, r)$ tensor. The anti-symmetric property is facilitated by the wedge product $\wedge$. In the simplest case, let us consider the two-form given by the wedge product of two one-forms:

$$\begin{aligned} dx^{i_1} \wedge dx^{i_2} &= -dx^{i_2} \wedge dx^{i_1}, \\ &= \left(dx^{i_1} \otimes dx^{i_2} - dx^{i_2} \otimes dx^{i_1}\right), \end{aligned} \tag{1.3.2}$$

where $dx^{i_1} \otimes dx^{i_2}$ is the tensor product of dx^{i_1} and dx^{i_2}.

This result may similarly be generalised to r-forms from r one-forms:

$$dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_r} = \sum_P \operatorname{sgn}(P) dx^{i_{P(1)}} \otimes dx^{i_{P(2)}} \otimes \ldots \otimes dx^{i_{P(r)}}, \tag{1.3.3}$$

P ranges over all permutations $\{1, \ldots, n\}$ and $\operatorname{sgn}(P)$ is the signature of the permutation,

$$\operatorname{sgn}(P) = \begin{cases} +1, & \text{if even permutation,} \\ -1, & \text{if odd permutation.} \end{cases} \tag{1.3.4}$$

The totally anti-symmetric property also ensures that should any two indices be the same, the wedge product is trivially zero.

1.3.1.1 The Exterior Product

Let us denote the vector space of r-forms at $p \in M$ by $\Omega_p^r(M)$, the set of r-forms in Eq. (1.3.3) forms a basis for $\Omega_p^r(M)$ and an element $\tilde{\omega} \in \Omega_p^r(M)$ can be expressed as

$$\tilde{\omega} = \frac{1}{r!}\, \omega_{i_1 i_2 \ldots i_r} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_r}, \tag{1.3.5}$$

where $\omega_{i_1 i_2 \ldots i_r}$ is also totally anti-symmetric. Zero-forms are trivially the real numbers, $\Omega_p^0(M) = \mathbb{R}$, while $\Omega_p^1(M) = T_p^*(M)$.

The wedge product between an arbitrary r-form $\tilde{\alpha}$ and q-form $\tilde{\beta}$ is

$$\tilde{\alpha} \wedge \tilde{\beta} = (-1)^{rq}\, \tilde{\beta} \wedge \tilde{\alpha}. \tag{1.3.6}$$

Thus the wedge product is defined to be the totally anti-symmetric multiplication operation for elements in $\Omega_p^n(M) : \Omega_p^r(M) \times \Omega_p^q(M) \to \Omega_p^{r+q}(M)$ which is a graded structure. We may then define an algebra (the exterior algebra) on M:

$$\Omega_p^*(M) \equiv \Omega_p^0(M) \oplus \Omega_p^1(M) \oplus \Omega_p^2(M) \oplus \ldots \oplus \Omega_p^n(M), \tag{1.3.7}$$

where $\Omega_p^n(M)$ is also known as the top-form, with $n = \dim M$.

1.3.1.2 The Exterior Derivative

Let us further define the operator d that maps a r-form to $(r+1)$-form,

$$\Omega_p^r(M) \xrightarrow{d} \Omega_p^{r+1}(M). \tag{1.3.8}$$

Its action on an r-form is given formally by

$$d\tilde{\omega} = \frac{1}{r!}\left(\frac{\partial}{\partial x^y}\omega_{i_1 i_2 \ldots i_r}\right) dx^y \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_r}. \tag{1.3.9}$$

It is a linear operation that follows Leibniz's[4] rule, in which given an r-form $\tilde{\alpha}$ and q-form $\tilde{\beta}$,

$$d\left(\tilde{\alpha} \wedge \tilde{\beta}\right) = d\tilde{\alpha} \wedge \tilde{\beta} + (-1)^r\, \tilde{\alpha} \wedge d\tilde{\beta}. \tag{1.3.10}$$

Also, due to the symmetry in partial derivatives $d^2 = 0$:

$$\begin{aligned} d^2\tilde{\omega} &= \frac{1}{r!}\left(\frac{\partial^2}{\partial x^\nu \partial x^\lambda}\omega_{i_1 i_2 \ldots i_r}\right) dx^\nu \wedge dx^\lambda \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_r}, \\ &= \frac{1}{r!}\left(\frac{\partial^2}{\partial x^\nu \partial x^\lambda}\omega_{i_1 i_2 \ldots i_r}\right) dx^\lambda \wedge dx^\nu \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_r}, \\ &= -\frac{1}{r!}\left(\frac{\partial^2}{\partial x^\nu \partial x^\lambda}\omega_{i_1 i_2 \ldots i_r}\right) dx^\nu \wedge dx^\lambda \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_r} = 0. \end{aligned} \tag{1.3.11}$$

The exterior derivative hence induces a sequence of maps

$$\{0\} \xrightarrow{i} \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \ldots \xrightarrow{d_{r-1}} \Omega^r(M) \xrightarrow{d_r} \Omega^{r+1}(M) \ldots \xrightarrow{d_{n-1}} \Omega^n(M) \xrightarrow{d_{n+1}} \{0\}, \tag{1.3.12}$$

where i is the inclusion map $0 \hookrightarrow \Omega^0(M)$[5] and $\dim M = n$.

The subscript in $\mathrm{im}(d_{r-1})$ and $\ker(d_r)$ helps denote the instance in the sequence at which the operation is carried out. This sequence generated by the exterior derivative is also called the de Rham complex.

1.3.1.3 de Rham Cohomology

Given the sequence of maps defined in Eq. (1.3.12), for $\Omega^r(M) \xrightarrow{d_r} \Omega^{r+1}(M)$

$$\begin{aligned} \mathrm{im}(d_r) &\subseteq \Omega^{r+1}(M) := \{d\tilde{\omega} | \tilde{\omega} \in \Omega^r(M)\}, \\ \ker(d_r) &\subseteq \Omega^r(M) := \{\tilde{\omega} \in \Omega^r(M) | d\tilde{\omega} = 0\}. \end{aligned} \tag{1.3.13}$$

Since $d^2 = 0$, $\mathrm{im}(d_{r-1}) \subseteq \ker(d_r)$. An r-form, $\tilde{\omega} \in \Omega^r(M)$ is closed if $\tilde{\omega} \in \ker(d_r)$, i.e. $d\tilde{\omega} = 0$; an r-form is exact if $\tilde{\omega} \in \mathrm{im}(d_{r-1})$, i.e. $\tilde{\omega} = d\tilde{\alpha}$, $\tilde{\alpha} \in \Omega^{r-1}(M)$.

The knowledge of this then allows us to define the quotient space

$$H^r_{\text{de Rham}} = \frac{\ker(d_r)}{\mathrm{im}(d_{r-1})}, \tag{1.3.14}$$

[4] Gottfried Wilhelm (von) Leibniz (1 July 1646 - 14 November 1716)

[5] The inclusion map simply means that 0 is an element in $\Omega^0(M)$ although not obtained from d.

or the rth de Rham cohomology group of M. This identifies closed forms that are not exact.

The study of the de Rham cohomology group in topology essentially helps to determine the nature of a boundary expressed as a differential form; the boundary of a boundary being the null set[6]. In supersymmetric theories, the space of differential forms are shown to be the natural representation of the fermionic fields, or Grassmannian numbers, when quantised. We will see this in Section 3.3.2.

1.3.1.4 Duality Transformations (Hodge Star)

In order to define the inner product of r-forms on M, we introduce the Hodge star operation. The dimension of the space of r-forms $\Omega^r_p(M)$, due to the anti-symmetric nature of r-forms is given by

$$\dim\left(\Omega^r_p(M)\right) = \binom{n}{r} = \binom{n}{n-r} = \frac{n!}{(n-r)!\,r!}, \tag{1.3.15}$$

where $\dim M = n$. This suggests that there is a duality between $\Omega^r_p(M)$ and $\Omega^{n-r}_p(M)$. Let us denote the duality transformation operation, or linear map from r-forms to $n-r$-forms by

$$\star : \Omega^r_p(M) \to \Omega^{n-r}_p(M). \tag{1.3.16}$$

Such a transformation is naturally an isomorphism, given that M is equipped with a metric g (a Riemannian manifold for example). Consider $dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_r}$ as the basis vector of $\Omega^r_p(M)$, then the $\star$ operation is defined as

$$\star\left(dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_r}\right) = \frac{\sqrt{|g|}}{(n-r)}\epsilon^{i_1\ldots i_r}{}_{j_{r+1}\ldots j_n} dx^{j_{r+1}} \wedge dx^{j_{r+2}} \wedge \ldots \wedge dx^{i_n}. \tag{1.3.17}$$

1.3.1.5 The Adjoint of the Exterior Derivative

Lastly, using the Hodge star operation, we may formally define the adjoint operation to the exterior derivative, $d^\dagger$. The adjoint of the exterior derivative maps r-forms to $r-1$-forms:

$$\Omega^r_p(M) \xrightarrow{d^\dagger} \Omega^{r-1}_p(M), \tag{1.3.18}$$

and it's defined as follows (for a Riemannian manifold)

$$d^\dagger = (-1)^{mn+m+1} \star d \star \tag{1.3.19}$$

A r-form $\tilde{\omega} \in \ker(d^\dagger)$, i.e. $d^\dagger\tilde{\omega} = 0$ is said to be co-closed. Similarly, $\left(d^\dagger\right)^2 = 0$.

[6]An example can be made by considering the boundary of a solid sphere. The boundary of a solid sphere is the two-dimensions surface and boundary of the two-dimensions surface is the null set.

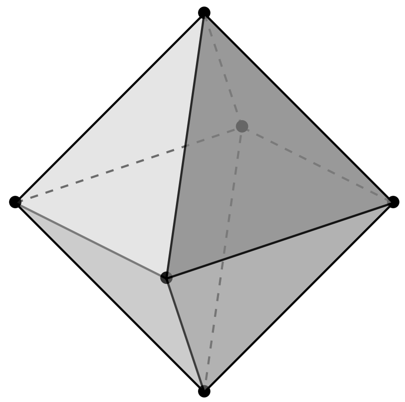


Figure 1.2: A polyhedron, with faces, edges and points indicated: we may consider the respective two-dimensional, one-dimensional and zero-dimensional objects as boundaries on the polyhedron. Shown here is an octahedron.

$$\begin{tikzcd} \Omega^{n-r}(M) \arrow[r, "(-1)^{mn+m+1}\, d"] \arrow[d, "\star"'] & \Omega^{n-r+1}(M) \arrow[d, "\star"] \\ \Omega^{r}(M) \arrow[r, "d^{\dagger}"'] & \Omega^{r-1}(M) \end{tikzcd}$$

Figure 1.1: Relations between the exterior derivatives and Hodge star in the exterior algebra.([4])

1.3.2 Euler Characteristic, Betti Numbers and the Euler-Poincare Theorem

In this section, we define and develop some relations between the Betti numbers of a topological space and its Euler characteristic. This will be necessary when we investigate Morse Theory, particularly for the Morse inequalities which will be shown in Section 1.3.4.

Consider a three-dimensional object: the polyhedron is a geometrical object surrounded by faces (or planes) which may be generalised to consider lines (edges), points (vertices) as boundaries. It is well known that the Euler characteristic of a general polyhedron $|K|$ is a topological invariant given by

$$\chi(K) = (\text{number of vertices in K}) - (\text{number of edges in K}) + (\text{number of faces in K})\,. \tag{1.3.20}$$

The Euler characteristic can be generalised to include smooth manifolds by associating the space to a homeomorphic polyhedron ('polyhedronisation' of space). More importantly, the result in Eq. (1.3.20) may be generalised to consider a n-dimensional polyhedron K:

$$\chi(K) \equiv \sum_{r=0}^{n} (-1)^r \, I_r, \tag{1.3.21}$$

where I_r is the number of r-simplixes in K. While the number of r-simplexs in K is equal to the number of r-forms on the n-dimensional manifold M, and hence,

$$\chi(K) \equiv \sum_{r=0}^{n} (-1)^r \dim \Omega^r(M) = \sum_{r=0}^{n} (-1)^r \, b_r(K). \tag{1.3.22}$$

The second equality is given by the Euler-Poincaré Theorem, where $b_r(K)$ is the rth Betti number and is defined by

$$b_r(K) \equiv \dim H_r \left(K; \mathbb{R}\right). \tag{1.3.23}$$

The Betti numbers are topological invariants, meaning they are considered under homeomorphisms; formally, the definition means that the rth Betti number is the rank of the rth homology group $H_r(K; \mathbb{R})$. Informally, the Betti number is the maximum number of cuts that can be made without dividing a surface into two separate pieces. Perhaps, it would be easier to appreciate the Betti numbers via an example: consider the torus T^2 with Betti numbers given by

$$b_0(K) = 1, \qquad b_1(K) = 2, \qquad b_2(K) = 1. \tag{1.3.24}$$

The rules for determining the Betti numbers of a topology may be summarised as follows:

- b_0 is the number of connected components,
- b_1 is the number of one-dimensional or 'circular' holes (S^1),
- b_2 is the number of two-dimensional 'voids' or 'cavities' (S^2).

Observe in Fig. 1.3 (a simply connected manifold) that there are two distinct one-dimensional holes, S^1, that can be drawn on the the torus ($b_1 = 2$) and a central void of the torus ($b_2 = 1$). Having $b_1 = 2$, is linked to the fact that a torus is homeomorphic to the Cartesian product of two circles: $S^1 \times S^1$.

Continuing with our discussion, by extension from the isomorphism between the homology and cohomology groups[7],

$$\chi(K) = \sum_{r=0}^{n} (-1)^r \, b^r(K) = \sum_{r=0}^{n} (-1)^r \dim H^r(M), \tag{1.3.25}$$

where $\dim H^r(M)$ denotes the rth cohomology group of the manifold M, which would be the de Rham comology.

The details are terse here and only introduced for quick reference in the later proofs of Morse Theory in supersymmetric quantum mechanics. For more details, refer to [4]

[7] The interested reader may find the proof in [4].

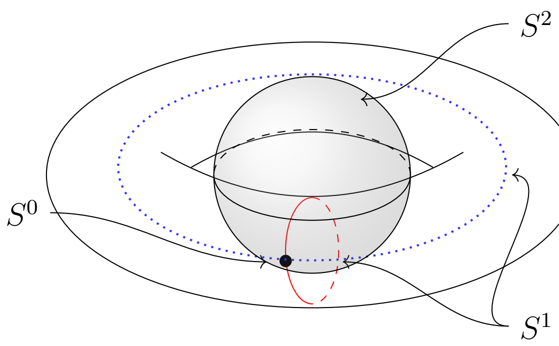


Figure 1.3: The calculation of the Betti numbers of the Torus T^2 is facilitated by observing the number of cuts made by S^r for the corresponding rth Betti number.

1.3.3 Laplace-Beltrami Operator, Harmonic Forms and Hodge Theorem

Using our definitions of the exterior and adjoint exterior derivatives in Sections 1.3.1.3 and 1.3.1.5, we define the generalisation of the Laplacian on differential forms: the Laplace-Beltrami operator.

Definition. The Laplace-Beltrami operator defined as the map from r to r-forms

$$\Delta : \Omega^r(M) \to \Omega^r(M), \tag{1.3.26}$$

it hence defines an automorphism between the space of differential forms. Formally, it is defined in terms of the exterior and adjoint exterior derivatives as

$$\Delta = \{d, d^\dagger\} = \{d\, d^\dagger + d^\dagger\, d\}. \tag{1.3.27}$$

A harmonic r-form is defined to satisfy the relation

$$\Delta\tilde{\omega} = 0, \tag{1.3.28}$$

this suggests that harmonic forms are both closed, $d\tilde{\omega} = 0$ and co-closed, $d^\dagger\tilde{\omega} = 0$ on M. Hodge's theorem states that the rth de Rham cohomology group $H^r(M, g)$ is isomorphic to the set of harmonic r-forms $\mathcal{H}^r(M, g)$ for a given compact, orientable Riemannian manifold:

$$H^r(M, g) \cong \mathcal{H}^r(M, g). \tag{1.3.29}$$

In particular, this suggests that we can re-express the Euler characteristic (topological structure) of a manifold to the harmonic forms (geometric structures) on the manifold:

$$\chi(M) = \sum_{r=0}^{n} (-1)^r\, b^r(K) = \sum_{r=0}^{n} (-1)^r \dim \mathcal{H}^r(M, g). \tag{1.3.30}$$

1.3.4 Morse Theory

Morse Theory will play an important part in our analysis in Section 3.2 in which what are known as superpotentials in a supersymmetric model would take on the interpretation of a Morse function on the target space. For now we will briefly describe Morse Theory and refer the interested reader to [5] for a more detailed study on Morse Theory. Critically, what Morse Theory is to the study of the topology of a manifold M is the ability to classify the topology by examining the functions defined on M.

The Morse function $f : \mathcal{M} \to \mathbb{R}$ is a differentiable real-valued function on a smooth, compact manifold $\mathcal{M}$ which has no degenerate critical points. What is meant by non-degenerate is that the Hessian of the Morse function f is non-singular.

$$
\begin{aligned}
H(h) &= \begin{pmatrix} \frac{\partial^2 h}{\partial {x^2}_1} & \frac{\partial^2 h}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 h}{\partial x_1 \partial x_n} \\ \frac{\partial^2 h}{\partial x^2 \partial x^1} & \frac{\partial^2 h}{\partial {x^2}_2} & \cdots & \frac{\partial^2 h}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 h}{\partial x_n \partial x_1} & \frac{\partial^2 h}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 h}{\partial {x}_n^2} \end{pmatrix}, \\
H(h)_{ij} &= \frac{\partial^2 h}{\partial x_i \partial x_j}.
\end{aligned} \tag{1.3.31}
$$

One of the conclusions of Morse Theory is that most functions are Morse functions [5, 16]. By examining the number and type of critical points of the Morse function h on M, we would be able to classify the topology of the manifold. This provides an alternative pathway to understanding complicated manifolds. What Morse Theory then illustrates is the parallel understanding of the topology similar to the understanding of the homology and cohomology groups on M, as well as the Betti numbers.

In [16], Nicolaescu describes the Morse Theory aptly as a ‘slicing’ technique. Starting from introducing an appropriate standard (function) as basis to decide where and how to ‘slice’ a manifold, quantifiable information can be extracted from the features that we observe as we rebuild the manifold by stacking the pieces back together. In order to illustrate his idea, let us look consider a torus T^2 tangent to plane V at point p, as shown in Fig. 1.4 below. Let us define the height function that gives the height above plane V to be $f : \mathcal{M} \to \mathbb{R}$. The height function is identified here as the Morse function. Furthermore, let M^a be the set of all points $x \in \mathcal{M}$ such that $f(x) \leq a$. We may then make the following statements from observing the individual pieces and rebuilding the manifold from ground up:

(1) If $a < 0$, then $M^a = \{0\}$,

(2) If $f(p) < a < f(q)$, M^a is a disk, homeomorphic to a point (0-cell),

(3) If $f(q) < a < f(r)$, M^a is homeomorphic to a cyclinder,

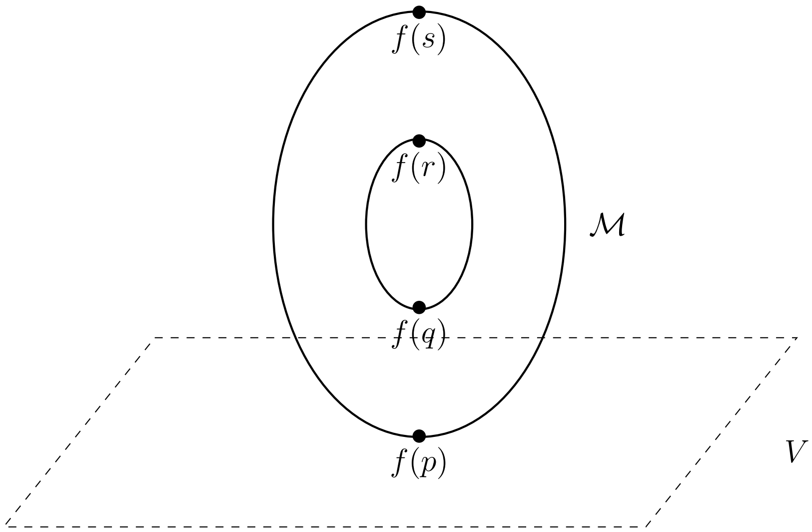


Figure 1.4: Let $y \in \{p, q, r, s\}$ be critical points on torus $\mathcal{M} = T^2$. And $\mathcal{M}$ is tangent to plane V at point p. $f : \mathcal{M} \to \mathbb{R}$ where $f(y)$ corresponds the height above plane V. ([5])

(4) If $f(r) < a < f(s)$, M^a is homeomorphic to a compact manifold of genus one having a circle as boundary.

(5) If $f(a) < a$, then M^a is the full torus T^2.

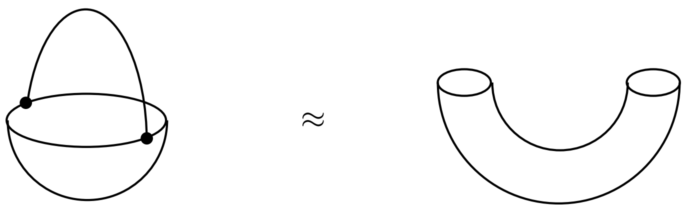
On hindsight, we will considering the homotopy type as we reconstruct the torus and the observe the changes as the height function passes through the various checkpoints $f(p), f(q), f(r), f(s)$ (critical points.

A homotopy type is a relaxed conditional equivalence relation. While a homeomorphism is restricted to a mapping f between topological spaces (X_1 and X_2) in which the inverse f^{-1} exists, we relax the condition on the existence of the inverse mapping when defining the homotopy type between two topological spaces. The two topological spaces are hence 'of the same homotopy type' [4].

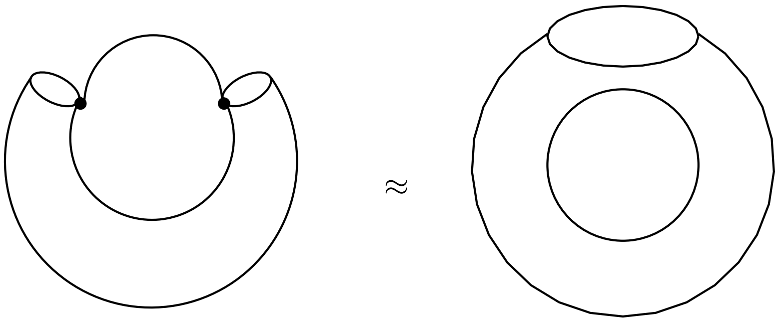
(1) $\to$ (2) is the operation of attaching a 0-cell onto the null set:

where "$\approx$" means that the two spaces are homotopy equivalent.

(2) $\to$ (3) is the operation of attaching a 1-cell:



Hence, M^a is homotopy equivalent to a cylinder $S^1 \times \mathbb{R}$. (3) $\rightarrow$ (4) is again the operation of attaching a 1-cell:



(4) $\rightarrow$ (5) is then the operation of attaching a 2-cell (disk) which then makes M^a a full torus.

The discussion establishes a relationship between the "attaching a n-cell" and the nature of the critical points. For the case of a torus, the critical points may be classified into three classes $f(p)$ corresponds to a basin (or a minimal point, from elementary calculus), $f(q)$ and $f(r)$ the passes (or saddle point) and lastly, $f(s)$ the peak (or maxima). We may then associate to each class an index 0, 1 and 2 respectively for the basin, pass and peak: this index is the Morse index. Notice that the change between M^a by "attaching a n-cell" corresponds quite nicely with the Morse index. Hence, by observing the cellular decomposition of the manifold M, it appears that we may infer the Morse index of its non-degenerate critical points. Vice-versa, the critical points of a function f on M allows us to infer the topology of the manifold.

Intuitively, the Morse index corresponds to the number of directions in which the function f decreases. The Morse index of a non-degenerate critical point correspond to the dimension of the largest subspace tangent to M at critical point y for which the Hessian is negative definite. In simpler terms, the Morse index corresponds to the number of negative eigenvalues at a critical point. Given a non-degenerate critical point y of $f : M \rightarrow \mathbb{R}$, the Morse lemma further suggests that there exists a chart $(x_1, x_2, \ldots, x_n)$ in a neighbourhood U centred at y, where no other non-degenerate critical points exist. The Morse function may then be expanded quadratically about y as follows

$$f(x) = f(y) - x_1^2 - \ldots - x_\mu^2 + x_{\mu+1}^2 + \ldots x_n^2, \tag{1.3.32}$$

where μ is the Morse index of y – this is the Morse Lemma. It can be easily seen that the critical points are indeed isolated from Eq. (1.3.32) by setting the first derivative to zero: we find that the critical point only exists at the origin hence completing the proof.

1.3.4.1 The Morse Inequalities

Lastly, in this review, we will explore the proof of what are known as the weak and strong Morse inequalities as given in [1]. The (abstract) *strong Morse inequality* is given by

$$\sum M_r t^r - \sum b_r t^r = (1+t) \sum Q_r t^r, \tag{1.3.33}$$

where $Q_r \geq 0$.

The (topological) Morse inequalities compares the number of critical points, M_r in a Morse function f with Morse index r and the b_r is the rth Betti number defined previously in Section 1.3.2.

We can rewrite the equality above using the identity

$$(1+t)^{-1} = \sum_{n \geq 0} (-1)^n t^n, \tag{1.3.34}$$

and hence we can deduce that

$$\sum_{r \geq 0} (-1)^r M_{n-r} - \sum_{r \geq 0} (-1)^r b_{n-r} = Q_n \geq 0, \tag{1.3.35}$$

which is equivalent to the abstract form of the strong Morse inequalities.

We can also see from Eq. (1.3.33) that it implies the *weak Morse inequalities*:

$$M_r \geq b_r. \tag{1.3.36}$$

According to Witten in [1], Eq. (1.3.33) is equivalent to the assertion that the critical points corresponds to the cohomology of the manifold[8]. This may be illustrated by considering the following: for every r, $r = 1, 2, \ldots, n$, let V_r be the vector space of dimension M_r. V_r is hence the vector space of critical points with Morse index r. Given this conditions, what Eq. (1.3.33) then suggests is that there exists a coboundary operator $\delta : V_r \to V_{r+1}$, that is nilpotent. The Betti numbers associated with the cohomology of δ then equal those of the manifold M. Hence, if we are able to find such a coboundary opeator, then the Morse inequalities must necessarily be satisfied. In Section 3.4 we will see that this coboundary operator exists as the exterior derivative of our quantised theory of the supersymmetric non-linear sigma model in one dimension. The interested reader may refer to [1] for Witten's proof of the Morse inequalities.

1.4 The Quantum Field Theory Formalism

The motivation to study quantum field theory arises when we consider systems at high energies as in particle physics. While relativistic quantum mechanics is a theory to describe the dynamics of a single particle, at high enough energies, particles creation and

[8]The proof of this assertion can be found in [16].

annihilation can occur and particle number is thus not conserved. Hence, to describe a relativistic system of many particles we need to develop a field theory. A quantum field theory is essentially the quantisation of a classical field theory and there is no one correct quantisation scheme in quantum field theory. Amongst the various possible quantisation schemes, we will restrict our discussion to the canonical quantisation and the path integral formalism.

In the typical structure of a quantum field theory, the problem is essentially solved if we are able to obtain the energy spectrum of the Hamiltonian defined for the theory. Evaluating the energy spectrum is however difficult in gauge theories where gauge fixing is a non-trivial problem[9]. Furthermore, unphysical results such as the existence of infra-red (IR) and ultra-violet (UV) divergences are also a problem for various quantum field theories. Here, we briefly outline the general formalism of quantum field theories.

1.4.1 Elements of Quantum Field Theory

Let us briefly discuss what are some of the essential elements in the study of a quantum field theory. Similar to the classical field theory, the quantum field theory describes a system with an infinite number of degrees of freedom. In field theory, we assign at least one degree of freedom to each point in space at some time t; this would be equivalent to taking a screenshot or a photo of the night sky. Now if we were to consider the dynamics of the system, we would need to compile multiple photos together at different instances. To further complicate matters, the information of the system can come in various forms (different colours); this would then make the degrees of freedom of our system become extremely large (or infinite). In this picture, our universe is four dimensions and the observation of the 'data' is similarly four-dimensional (three to triangulate the position of a star/source, another for time). In this case, we say that the *base manifold* (the universe that describes our system) is four-dimensional.

Fields and Target Spaces

Next, we need to describe the kind of data that is being collected – these data are represented as *fields*. Fields can come in different forms; for the simplest case we can choose the fields to be scalar functions $\phi : M \to \mathbb{R}$ or $\mathbb{C}$. More generally, this can be extended to consider a more complicated *target space* and hence gives something like $\phi : M \to N$, where N is the target space where we actually interpret the data (the photos). Scalar fields would correspond to the intensity of light emitted per unit square area in space, or the temperature distribution in a room and the target space would be our camera sensor or photos themselves.

[9] An example can be made of Quantum Electrodynamics in which there is a freedom to choose between, for example, the Lorentz gauge or the Coloumb gauge.

The Action

In order to make comparisons between the different temperature distributions in a room over the course of some time, we assign a value to each configuration in spacetime. The action is the functional that takes in the scalar fields which are functions themselves to give us a corresponding value. It can be expressed mathematically as $S : \mathcal{C} \to \mathbb{R}$ where $\mathcal{C}$ is the space of configuration. In field theories, the action usually takes the form of $S = \int d^4x\mathcal{L}$ where $\mathcal{L}$ is known as the Lagrangan or Lagrangian density. It is the expression from which the dynamics of the system is represented.

1.4.2 Lorentz Invariant Lagrangian

The starting point of realistic quantum field theories is outlining a Lorentz invariant Lagrangian, or a Lagrangian density to be specific. Just as in Relativity, a Lorentz invariant Lagrangian impose that our results are not dependent on the frame of reference. Typically, as in the case of the Klein-Gordon equation, the form of the Lagrangian is further constrained to give physical equations of motion which obey the mass-shell condition[10],

$$-E^2 + p^2c^2 = -m^2c^4. \tag{1.4.1}$$

The Lagrangian is a function that takes in the fields and their derivatives as arguments. Interaction terms can be introduced when investigating interaction theories with coupling constants, λ, to represent the strength of the interaction. Hence depending on the theory to be examined, the Lagrangian is then specified.

As an example, we state the Lagrangian of the scalar ϕ^4 theory that describes the Higgs mechanism:

$$\mathcal{L}(\phi, \partial_\mu\phi, \lambda) := \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{1.4.2}$$

where the last term represents self-interactions. In a free field theory, the last term is dropped.

1.4.3 Principle of Stationary Action

Following the Lagrangian, we can obtain the equations of motion for the theory. This is most easily obtained for the non-interacting, free field theory situation. This is commonly accomplished by applying the variational principle for the action, the principle of stationary action or Hamilton's Principle. This may be expressed as

$$\delta S = \delta\left(\int dx^4\,\mathcal{L}\right) = 0, \tag{1.4.3}$$

from which we may obtain the Euler-Lagrange equations of the following form:

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\right) = 0. \tag{1.4.4}$$

[10]We have chosen the convention for the Minkowski metric to take the form of (-1,1,1,1)

Considering the free field theory for Eq. (1.4.2) in which $\lambda = 0$, this gives

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0. \tag{1.4.5}$$

$\lambda = 0$ when corresponds to a free theory without self-interactions. This is recognised as the Klein-Gordon equation that describes the Higgs boson, the only scalar particle in the Standard Model.

1.4.4 Symmetries

We may also further identify the various generators of symmetry in the theory in the Lagrangian formalism. This may be accomplished by taking the variation of the action/Lagrangian, or looking for internal symmetries. An example of an internal symmetry would be to look for $U(1)$ gauge invariance of the kind that generalises the Lagrangian of a single scalar field formulated in 1.4.2. We may do so by extending Eq. (1.4.2) for two scalar fields ϕ_1 and ϕ_2, expressed in complex variables,

$$\mathcal{L} := \partial_\mu \overline{\phi} \partial^\mu \phi - m^2 |\phi|^2 - \lambda |\phi|^2, \tag{1.4.6}$$

where we've introduced $\phi = \phi_1 + i\phi_2$ and $\overline{\phi}$ its complex conjugate.

To show that the Lagrangian is invariant under $U(1)$ gauge symmetry, consider the following transformations:

$$\phi \to e^{i\gamma} \phi, \tag{1.4.7}$$

$$\overline{\phi} \to e^{-i\gamma} \phi. \tag{1.4.8}$$

One can show that the Lagrangian is indeed invariant for the given set of transformations (up to a total derivative term).

On the other hand, a Lorentz-invariant Lagrangian would be made up of Lorentz scalars: $\partial_\mu \overline{\phi} \partial^\mu \phi$, $|\phi|^2$. In fact, it's easier to check for Lorentz-invariance in the Lagrangian than to do so in the Hamiltonian formalism. This is because the Hamiltonian is typically not explicitly Lorentz invariant by construction. Take for example the Schrödinger's equation that is shown to have a first order derivative in time but second order in space. We say that the theory does not treat time and space on a equal footing and hence is not manifestly Lorentz invariant. We would however assume that a theory (or Hamiltonian) constructed from a Lorentz invariant Lagrangian remains so after manipulation given that the physical description of the system does not change. We can then also be certain that the theory remains Lorentz invariant when we have quantised the classical theory.

1.4.5 Hamiltonian

The Hamiltonian may be defined as follows for the bosonic variables and will take on a slightly different definition for the fermionic variables. We first define the conjugate momenta for the free field theory with equations of motion given by Eq. (1.4.5),

$$\Pi_\phi = \frac{\partial \mathcal{L}}{\partial \left(\partial_0 \phi\right)}, \tag{1.4.9}$$

which may be generalised for arbitrary number of fields in a given theory. Using this definition, via the Legendre transformation, we obtain the Hamiltonian (density) function in terms of the conjugate momenta and position (bosonic) variables:

$$\mathcal{H}(\Pi_\phi, \phi) = \Pi_\phi \dot{\phi} - \mathcal{L}. \tag{1.4.10}$$

For the free field theory we would arrive at

$$\mathcal{H}(\Pi_\phi, \phi) = \frac{1}{2}\Pi_\phi^2 + \frac{1}{2}\partial_i\phi\partial^i\phi + \frac{1}{2}m^2\phi^2. \tag{1.4.11}$$

We delay the definition for Hamiltonian for fermionic variables till Section 2.1.

1.4.6 The Partition Function

In d-dimensions then, the theory may be defined by a partition functional or path integral with Minkowski signature by

$$Z(J) = \int \mathcal{D}\phi \, e^{iS(\phi,J)} \tag{1.4.12}$$

where $\mathcal{D}\phi$ is an ill-defined integration measure in regular measure theory, with ϕ as the placeholder for all scalar fields and J being the placeholder for all currents.

We further notice that the weighted exponential is an oscillatory term and hence the convergence of the integral is questionable. The different paths/mappings are weighted equally but with varying phase, determined by the Hamilton's principle function, S. By imposing Hamilton's Principle, we will only consider weights for which the variation in the action $\delta S = 0$ (stationary paths/states), while paths that are highly oscillatory about the stationary paths will interfere destructively and not contribute to the path integral.

The path integral formalism, while having its own set of problems, is an alternative quantisation scheme to the canonical quantisation that has been shown to be equivalent [17, 18]. Some of the advantages of the path integral formalism are listed here:

(1) the quantum theory is Lorentz invariance by construction, enforced by an appropriate Lagrangian,

(2) avoids defining commutation relations of quantum operators,

(3) (as we will see) pertubation methods are easier to implement in the path integration approach without ordering issues.

Nevertheless, while the calculations may be generally more tedious, the path integral approach can be shown to be equivalent to the canonical quantisation[11]. We can also appreciate the path integral better from the perspective of differential geometry which is the angle this paper will take on.

In particle physics, calculations for scattering cross section and decay may be obtained by considering perturbation theory to the path integral and obtaining perturbative results to an appropriate order. The evaluation of such calculations are often tedious and are limited mostly to tree level calculations, facilitated by Feynman diagrams. Feynman diagrams are graphical representation of the interactions in an interacting theory obtained by considering the perturbative result.

While the Feynman diagrams have far reaching implications in particle physics, we will disregard this in our discussion. The interested reader may refer to any appropriate particle physics textbook. We are now ready to discuss the zero-dimensional supersymmetric quantum field theory in the following chapter, where we shall also elaborate on the path integral formalism in greater detail.

[11] We can find an example of this for the harmonic oscillator developed in Appendix A.1.

Chapter II

Zero-Dimensional SUSY QFT

Before we discuss the supersymmetric generalisation of the zero-dimensional quantum field theory, let us first look at some general results in a regular quantum field theory.

A zero-dimensional quantum field theory is a toy model where results of complicated theories can be extracted without dealing with the infinite degrees of freedom associated to the real world. The base manifold M would correspond to being a point. While this seems to be trivial, and would appear to be unphysical given that the universe does not look zero-dimensional, the calculations and results obtained for zero-dimensions may be rigorously explored. The results then provide a basis as we extend into higher dimensions (one-dimensional QFT and beyond). By restricting our theory (a toy model) to zero dimensions, the number of degrees of freedom is drastically reduced, hence simplifying our calculations.

With reference to the expression for the path-integral in Eq. (1.4.12) for arbitrary d-dimensions, we recall that the integration measure is ill-defined. Direct evaluation of the integral would require us to discretise the integral into finite time intervals to obtain a well-defined summation before extending the result to a continuous integral. Even then, mathematically this "recipe" is not rigorously well-defined despite its success in obtaining accurate results. Even without supersymmetry, the discussion of zero-dimensional QFTs provides a general result to determine the Feynman diagrams (graphs) and analysis of complicated theories such as gauge theory in higher dimensions.

It can be seen that for our zero-dimensional theory there will not be any formal quantisation. This is to be expected as it is impossible to define the conjugate momenta. The theory is hence neither a classical nor quantum formulation. However, we will see that the results do carry over into the one-dimensional theory.

For QFT in zero-dimensions, we will introduce the set of scalar fields x over the base manifold M. The scalar fields are hence defined to be smooth maps between the base manifold onto the target space $\mathbb{R}$. Using the Lagrangian formalism, the dynamics of the particle is described on the target space $\mathbb{R}$. This can be seen from Fig. 2.1.

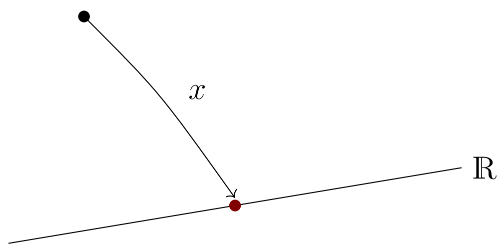


Figure 2.1: Map of $M = \{\text{pt}\}$ to $\mathbb{R}$ by the scalar field x; $x : \{\text{pt}\} \to \mathbb{R}$.

We would also identify $M = \{\text{pt}\}$ for the given theory, and hence for $d = 0$,

$$x : \{\text{pt}\} \to \mathbb{R}, \tag{2.0.1}$$

and the action $S[x]$ would be a function of the scalar field x[1].

It is not much of a field given that the parameter space (base manifold M) is a simply a point, even so we will see that our toy model will manifest some interesting results. Our discussion will be limited to scalar fields given that the Lorentz group for a zero-dimensional theory is trivial. This also means that the theory is necessarily Lorentz invariant.

We may then reintroduce the partition function discussed earlier in Eq. (1.4.12) but reformulated for zero-dimensions. The partition function here would be similar to what is used in statistical mechanics to describe an ensemble of particles, or rather the sum of all possible states for that ensemble given the Hamiltonian of the system and a well defined phase space description. The partition function described here has a similar interpretation: it is represented as

$$Z := \int dx e^{-S[x]/\hbar}, \tag{2.0.2}$$

where the partition function is the weighted integration over the space of all maps x, or the configuration space.

Let us define for compactness, the field configuration space to be $\mathcal{C}$, then $x \in \mathcal{C}$[2]. The action $S[x]$ would then simply be the map of $\mathcal{C} \to \mathbb{R}$. We can thus also see that if we take $S[x] \gg \hbar$, the classically accessible states have the greatest weight. This agrees with our common day intuition that classical behaviour is dominant over quantum states.

The correlation functions representing the expectation value for some particular configuration would hence be weighted integrals of the form,

$$\langle f(x) \rangle := \int dx f(x) e^{-S[x]/\hbar}. \tag{2.0.3}$$

Henceforth, we will set $\hbar = 1$ in our calculations. In zero-dimensions, the exponential weights are naturally Euclidean given that there's no spacetime structure to define for the zero-dimension manifold.

[1] Note that we previously used ϕ in Section 1.2. Use of the label x will be justified later in Chapter III.

[2] Typically, $\mathcal{C}$ would be a infinite dimensional and this is the key problem with quantum field theories. If suppose we are able to reduce it to a finite dimensional integral or a discrete sum, the problem would then be circumvented.

2.1 Fermionic Variables

In supersymmetric quantum field theories, the underlying principle is motivated by the existence of *fermionic variables* or *odd fields* ψ^a. These objects are also known to physicists as *Grassmann*[3] *variables*/numbers. Together, the bosonic and fermionic variables are defined on the base manifold M. The Grassmann variables being the most important feature to supersymmetric theories deserves a great amount of definition and in the following sections we will dwelve into the mathematical foundation of these objects.

Together, the two sets of variables form an associative and up to sign, commutative and anti-commutative algebra. There is a $\mathbb{Z}_2$ gradation that assigns to all the bosonic variables a $+1$ and to all the fermionic variables a -1, and is compatible with the multiplication in the algebra. Their (anti-)commutation relations are defined to be:

$$\left[x^i, x^j\right] = 0, \tag{2.1.1}$$

$$\left[x^i, \psi^a\right] = 0, \tag{2.1.2}$$

$$\{\psi^a, \psi^b\} = 0. \tag{2.1.3}$$

Note that Eq. (2.1.3) further implies that $(\psi^a)^2 = 0$.

Also, pairs of $\psi^a\psi^b$ are commutative or "pair-commutable", in that

$$\psi^a\psi^b\psi^c = \psi^b\psi^c\psi^a, \tag{2.1.4}$$

$$x^i\psi^a\psi^b = \psi^a\psi^b x^i, \tag{2.1.5}$$

behaving like our bosonic variables.

2.1.1 Grasmann Numbers

We interrupt our discussion to further qualify what are Grassmann variables and what is meant by a $\mathbb{Z}_2$ gradation. Due to the properties of the exterior algebra and its relation to *oriented* volume (see top-forms), its geometrical properties motivates our interest in this object, seeing how General Relativity is a theory of differential geometry.

The Grassmann numbers are defined as elements of the exterior algebra $\Lambda(V)$ over a vector space V. The set of ψ^a and x^i form the Grassmann algebra/numbers because it has the following decompositions

$$\Lambda(V) = \Lambda^0(V) \oplus \Lambda^1(V) \tag{2.1.6}$$

and that

$$\Lambda^i(V)\Lambda^j(V) \subseteq \Lambda^{i+j}(V), \tag{2.1.7}$$

where the subscripts are read modulo 2. These properties would suggests that $\Lambda(V)$ respects the group multiplication table for the cyclic group $\mathbb{Z}_2$, hence a $\mathbb{Z}_2$ graded algebra.

[3]Hermann Günther Grassmann (1809-1877)

The anti-commutation relations in Eq. (2.1.3) may also remind us of the γ matrices in the Dirac formulation of spinors, or more generally of the Clifford algebra. The Clifford algebra $C\ell(V, q)$ is defined for an element v in vector space V,

$$v^2 = q(v)\mathbb{1}, \tag{2.1.8}$$

where $q(v)$ denotes a quadratic form of $v \in V$. In the familiar case for Dirac spinors we have

$$\{\gamma_\mu, \gamma_\mu\} = 2\eta_{\mu\nu}\mathbb{1}. \tag{2.1.9}$$

For a more physical picture, we may say that the Clifford algebra is a quantisation of the exterior algebra (we may see this as a transition from Eq. (2.1.3) to Eq. (2.1.9)), or the Grassmann algebra. Where $q(v) = 0$, we would indeed see that the Clifford algebra reduces to Eq. (2.1.3).

It can be easily shown that the bosonic and fermionic variables obey Eqs. (2.1.6)–(2.1.7) from Eqs. (2.1.1)–(2.1.5). These properties in fact makes the Grassmann algebra a *superalgebra*. The superalgebra defined by the (anti-)commutation relations Eqs. (2.1.1)–(2.1.3) and is correspondingly a $\mathbb{Z}_2$ gradation.

We may identify the bosonic variables $x^i \in \Lambda^0(V)$ as 'even' and fermionic variables $\psi^a \in \Lambda^1(V)$ as 'odd' and define *parity* for the variables as below in Eqs. (2.1.10) and (2.1.11). They would hence have the homogeneous property that

$$|x^i| = 0, \tag{2.1.10}$$

$$|\psi^a| = 1, \tag{2.1.11}$$

where the parity of a variable is determined according to whether it is found in $\Lambda^0(V)$ or $\Lambda^1(V)$. Also, for $x, y \in \Lambda^i(V)$,

$$|xy| = |x| + |y|. \tag{2.1.12}$$

Take note that the parity defined is different from that of what is used to differentiate handedness in typical particle physics.

2.1.2 Berezinian Integrals

Lastly, the integration rules of the Grassman variables, also known as *Berezinian integrals*, are defined by

$$\int d\psi \; = 0, \int d\psi\, \psi = 1. \tag{2.1.13}$$

$$\int \psi^1 \dots \psi^n d\psi^1 \dots d\psi^n = 1. \tag{2.1.14}$$

Notice how the integrations are evaluated as though it's a derivative. The formulation of this branch of integrals, or superanalysis is mostly attributed to Felix A. Berezin[4] in

[4]Felix Alexandrovich Berezin (1931 - 1980)

[19]. Following our discussion here, we will only consider Grassmann even actions despite that this is not necessarily a constraint to be imposed. There is, however, some interest in actions that are Grassmann odd [20]. We will however not explore this in our review. The Berezinian integrals also forms a constraint on the differential forms of the Grassmann variables as will be seen when we define the Jacobian for a change of variables.

If we were to see the fermionic variables ψ_a as a representation of the fermionic super-partners to the bosonic particles in the Standard Model, then naturally we would interpret the non-commutativity defined in Eq. (2.1.3) as a basis for the spin statistics/Pauli exclusion principle. Of course, given that we have only defined this for zero dimensions, the full interpretation that establishes this relation is not apparent here. It will however be clearer in $(0+1)$ dimensions. It would also be interesting to investigate SUSY in higher dimensions to see the realisation of the variables as fermions in nature. Yet again, this will not be attempted in our review.

We consider the simplest case for which the action only has even pairs of independent fermionic variables,

$$S(\psi) = \frac{1}{2}\psi^i M_{ij}\psi^j. \tag{2.1.15}$$

The partition function is hence

$$Z = \int \prod_k d\psi^k e^{-\frac{1}{2}\psi^i M_{ij}\psi^j} = \mathrm{Pf}(M). \tag{2.1.16}$$

The Pfaffian (Pf) is a property of skew-symmetric matrices and is considered as a polynomial. It is non-vanishing only for $2n \times 2n$ skew-symmetric matrices, which is easily satisfied given that our action is Grassmann even. The Pfaffian further has the property of $\mathrm{Pf}(M)^2 = \det(M)$ where M is a $2n \times 2n$ skew-symmetric matrix. For the simple case of $n = 1$, $\mathrm{Pf}(M)^2 = |M_{ij}|^2$, for $i \neq j; i, j = 1, 2$.

Let us consider the next non-trivial action which includes both bosonic and fermionic variables of the form

$$S(x, \psi^1, \psi^2) = S_0(x) - \psi^1\psi^2 S_1(x), \tag{2.1.17}$$

where

$$S_0 = \frac{1}{2}(\partial h)^2 \text{ and } S_1(x) = \partial^2 h, \tag{2.1.18}$$

$$\text{for } h := h(x). \tag{2.1.19}$$

h is defined here to be a polynomial function of the real variable x; it is also known as the superpotential.

We further define that $\partial h := \dfrac{\partial h(x)}{\partial x}$. Explicitly, we are considering the zero-dimensional QFT defined for

$$S(x, \psi_1, \psi_2) := \frac{1}{2}(\partial h)^2 - \partial^2 h\, \psi_1\psi_2. \tag{2.1.20}$$

We pause for a moment to consider the action described in Eq. (2.1.17). Consider the partition function for this generalised action,

$$\begin{aligned} Z &= \int dx d\psi^1 d\psi^2 e^{-S_0+\psi^1\psi^2 S_1(x)}, \\ &= \int dx d\psi^1 d\psi^2 e^{-S_0}(1+\psi^1\psi^2 S_1(x)), \\ &= \int dx d\psi^1 d\psi^2 e^{-S_0} + \int dx d\psi^1 d\psi^2 e^{-S_0}\psi^1\psi^2 S_1(x)). \end{aligned} \tag{2.1.21}$$

Integrating over the fermionic variables,

$$Z = \int dx e^{-S_0} S_1(x)). \tag{2.1.22}$$

Interpreting this result suggests that it is possible to treat the second term in Eq. (2.1.17) as a source, and that the partition function here is in fact evaluating the correlation function of $S_1(x)$ with the "actual action" given by S_0. This picture that we have here would be useful later when we do the calculations and demystifies our results. This interpretation in fact justifies why the $S_0(x)$ is defined in Eq. (2.1.18) – $S_0(x)$ would necessarily be the action for a bosonic particle in non-supersymmetric theories.

Hence, the action defined here describes a particle with both bosonic and fermionic degrees of freedom: the bosonic scalar fields x_i defines the position of the particle while the fermionic fields are degrees of freedom defined on the base manifold. This follows from our definition that $x : M \to \mathbb{R}$. While the fermionic variables ψ^a are not defined to be mappings of this kind, they are not defined on the target manifold. Such an action exihibits supersymmetry in that one can exchange bosonic variables with fermionic variables while ensuring that the action remains the same.

The corresponding partition function for an action defined in Eq. (2.1.20) would be

$$Z = \int dx d\psi^1 d\psi^2 e^{-S_0+\psi^1\psi^2 S_1(x)}. \tag{2.1.23}$$

Even without evaluating the partition explicitly, we would expect that a partition function of this form would be zero from our interpretation of the SUSY action. If there are no critical points in $h(x)$ such that $\partial h(x) = 0$, the partition function should turn out to be zero referencing to our general result in Eq. (2.1.22).

Considering variation of the fields, the action transforms as $S \to S + \delta S$, with

$$\begin{aligned} x &\to x + \delta x, \\ \psi_i &\to \psi_i + \delta\psi_i. \end{aligned} \tag{2.1.24}$$

The variation in S can be expressed as

$$\delta S = (\partial h\, \partial^2 h - \frac{1}{2}\partial^3 h\, \psi_1\psi_2)\delta x - (\partial^2 h\, \psi_1 \delta\psi_2 - \partial^2 h\, \psi_2\delta\psi_1). \tag{2.1.25}$$

From Hamilton's Principle we'll be looking for variations in the fields such that $\delta S = 0$ or for δS to be expressed as a total differential, then expressed as a boundary term.

Furthermore, we wish to look for supersymmetry and this can be seen by considering the set of transformations that exchange bosonic with fermionic fields and vice-versa:

$$\begin{aligned} \delta_\epsilon x &= \epsilon^{(1)}\psi_1 + \epsilon^{(2)}\psi_2, \\ \delta\psi_1 &= \epsilon^{(2)}\partial h, \\ \delta\psi_2 &= -\epsilon^{(1)}\partial h. \end{aligned} \tag{2.1.26}$$

$\epsilon^{(i)}$ is introduced as an infinitesimal fermionic variation parameter. It is hence also a Grassmann odd variable and obeys the following algebra:

$$\begin{aligned} \{\epsilon^{(1)}, \epsilon^{(2)}\} &= 0, \\ \{\psi_1, \psi_2\} &= 0, \\ \{\psi_i, \epsilon^{(i)}\} &= 0. \end{aligned} \tag{2.1.27}$$

Transformations of these kinds, by exchanging bosonic field for fermionic fields, generated by odd parameters ($\epsilon^{(i)}$) are known as supersymmetries. We may check that under the transformations listed in Eq. (2.1.26), $\delta S = 0$. Substituting Eq. (2.1.26) into Eq. (2.1.25),

$$\begin{aligned} \delta S &= (\partial h\, \partial^2 h - \frac{1}{2}\partial^3 h\, \psi_1\psi_2)(\epsilon^{(1)}\psi_1 + \epsilon^{(2)}\psi_2) - [\partial^2 h\, \psi_1(-\epsilon^{(1)}\partial h) - \partial^2 h\, \psi_2(\epsilon^{(2)}\partial h)], \\ &= \partial h\partial^2 h\, (\epsilon^{(1)}\psi_1 + \epsilon^{(2)}\psi_2) - [-\partial^2 h\, \psi_1\epsilon^{(1)}\partial h - \partial^2 h\, \psi_2\epsilon^{(2)}\partial h], \\ &= \partial h\partial^2 h\{\epsilon^{(1)}, \psi_1\} + \partial h\partial^2 h\{\epsilon^{(2)}, \psi_2\}, \\ \therefore\, \delta S &= 0. \end{aligned} \tag{2.1.28}$$

We may also wish to show that the integration measure $dxd\psi_1 d\psi_2$ is invariant under the transformation, i.e. the Jacobian (or superdeterminant) of the transformation, $J = 1$.

$$J^{-1}(\hat{x}, \hat{\psi}_1, \hat{\psi}_2) = \begin{vmatrix} \frac{\partial\hat{x}}{\partial x} & \frac{\partial\hat{x}}{\partial\psi_1} & \frac{\partial\hat{x}}{\partial\psi_2} \\ \frac{\partial\hat{\psi}_1}{\partial x} & \frac{\partial\hat{\psi}_1}{\partial\psi_1} & \frac{\partial\hat{\psi}_1}{\partial\psi_2} \\ \frac{\partial\hat{\psi}_2}{\partial x} & \frac{\partial\hat{\psi}_2}{\partial\psi_1} & \frac{\partial\hat{\psi}_2}{\partial\psi_2} \end{vmatrix} = \begin{vmatrix} 1 & \epsilon^{(1)} & \epsilon^{(2)} \\ \epsilon^{(2)}\partial^2 h & 1 & 0 \\ -\epsilon^{(1)}\partial^2 h & 0 & 1 \end{vmatrix}, \tag{2.1.29}$$

$$\begin{aligned} &= -\epsilon^{(1)}\partial^2 h\, (-\epsilon^{(2)}) + 1 - \epsilon^{(1)}\epsilon^{(2)}\partial^2 h \\ &= 1 = J(\hat{x}, \hat{\psi}_1, \hat{\psi}_2). \end{aligned}$$

2.2 Localisation and Supersymmetry

We are now ready to explore an important characteristic of supersymmetric theories in general: the localisation principle. It provides us with a way to rigorously evaluate the partition function/path integral and correlation functions.

Let us consider a more specific example of Eq. (2.1.26) in which we let $\epsilon^{(1)} = \epsilon^{(2)} = -\psi_1/\partial h$, valid for $\partial h \neq 0$. Eq. (2.1.26) would then take the following form:

$$\begin{aligned}
\delta_\epsilon x &= -\frac{\psi_1\psi_2}{\partial h}, \\
\delta\psi_1 &= -\psi_1, \\
\delta\psi_2 &= +\psi_1.
\end{aligned} \tag{2.2.1}$$

We hence consider a change of variables

$$\begin{aligned}
\hat{x} &:= x - \frac{\psi_1\psi_2}{\partial h}, \\
\hat{\psi}_1 &:= \alpha(x)\psi_1, \\
\hat{\psi}_2 &:= \psi_1 + \psi_2.
\end{aligned} \tag{2.2.2}$$

where $\alpha(x)$ is some arbitrary function of x. From our previous result we would expect that

$$S(x, \psi_1, \psi_2) = S(\hat{x}, \hat{\psi}_1 = 0, \hat{\psi}_2 = 0) = \frac{1}{2}(\partial h(\hat{x}))^2. \tag{2.2.3}$$

From this, it seems that via the transformations in Eq. (2.2.1) the "new" superpartners have decoupled from our "new" bosonic variable. Hence, we can see from the action given here that the corresponding partition function would possibly be zero if the Jacobian for this transformation does not have terms of the form $\hat{\psi}_1\hat{\psi}_2$ before integrating over the fermionic fields.

Having said that, we will now determine the Jacobian for the given change of variables. The integration measure in terms of the new variables can be found by considering the Jacobian of the change of variables as follows[5]:

$$\begin{aligned}
J(x, \psi_1, \psi_2) = \begin{vmatrix} \frac{\partial x}{\partial \hat{x}} & \frac{\partial x}{\partial \hat{\psi}_1} & \frac{\partial x}{\partial \hat{\psi}_2} \\ \frac{\partial \psi_1}{\partial \hat{x}} & \frac{\partial \psi_1}{\partial \hat{\psi}_1} & \frac{\partial \psi_1}{\partial \hat{\psi}_2} \\ \frac{\partial \psi_2}{\partial \hat{x}} & \frac{\partial \psi_2}{\partial \hat{\psi}_1} & \frac{\partial \psi_2}{\partial \hat{\psi}_2} \end{vmatrix} &= \begin{vmatrix} \frac{1 - \hat{\psi}_1\hat{\psi}_2\partial^2 h(\hat{x})}{\alpha(\hat{x})\left(\partial h(\hat{x})\right)^2} & \frac{\alpha(\hat{x})\hat{\psi}_2}{\partial h(\hat{x})} & \frac{-2\hat{\psi}_1}{\partial h(\hat{x})} \\ 0 & \alpha(\hat{x}) & 0 \\ 0 & -\alpha(\hat{x}) & 1 \end{vmatrix} \\
&= \alpha(\hat{x}) - \hat{\psi}_1\hat{\psi}_2(\frac{\partial^2 h(\hat{x})}{\partial h(\hat{x})^2}).
\end{aligned} \tag{2.2.4}$$

Hence,

$$dx d\psi_1 d\psi_2 = \Big(\alpha(\hat{x}) - \hat{\psi}_1\hat{\psi}_2(\frac{\partial^2 h(\hat{x})}{\partial h(\hat{x})^2})\Big) d\hat{x} d\hat{\psi}_1 d\hat{\psi}_2, \tag{2.2.5}$$

[5]In order to evaluate the Jacobian, it is important to recall that there are constraints on integration of the fermionic variables (the Berezinian integrals), as defined by Eq. (2.1.13) found in Section 2.1.2.

and the partition function is expressed as

$$Z = \int d\hat{\psi}_1 \int d\hat{x} \int \hat{\psi}_2 e^{-S(\hat{x},0,\hat{\psi}_2)} \alpha(\hat{x}) - \int d\hat{x} \int \hat{\psi}_1\, d\hat{\psi}_1 \int \hat{\psi}_2\, d\hat{\psi}_2\, e^{-S(\hat{x},0,\hat{\psi}_2)} (\frac{\partial^2 h(\hat{x})}{\partial h(\hat{x})^2}). \tag{2.2.6}$$

The first term in the partition function vanishes because of the Berezinian integration rules Eq. (2.1.13), while the second term is zero because it can be expressed as a total differential in x, which is a boundary term that can be arbitrarily set to zero.

Following our understanding that this partition function is in actual fact evaluating the correlation function $S_1(x) = \partial^2 h(x)$, let us now consider the evaluation for which $\partial h(x) = 0$, i.e. the correlation function about the critical points of $h(x)$. Consider then $\partial h(x_c) = 0$, such that $\delta\psi_i = 0$ and terms where $\partial h(x) \neq 0$ does not contribute to the partition function/integral. That is, in the neighbourhood of points around which $\partial h = 0$, we are not able to perform the exchange of $x \to \psi_i$, or that these points are not 'supersymmetric'. It can be said that supersymmetry breaks down at these critical points. Another obvious consequence would be that our action is described by the fermionic term alone. So the action defined at these points would be

$$S(x_c, \psi_1, \psi_2) = \partial^2 h(x_c)\psi_1\psi_2. \tag{2.2.7}$$

The 'asymmetry' in our action is perhaps a hint at why supersymmetry breaks down at these points. This property is linked to the existence of instantons in our theory. We will leave the formal discussion on instantons to Section 3.4.

2.3 The Superpotential

It is interesting to examine the superpotential defined and the form of the action considered. As we can see, $h(x)$ does not appear explicitly in the action directly but instead, the action considers the derivatives of the superpotential. If we were to consider an action of the following form

$$S'(x, \psi_1, \psi_2) = \frac{1}{2}(f(x))^2 - \partial f\, \psi_1\psi_2, \tag{2.3.1}$$

by an equivalent definition with $f(x)$, we stand to lose some physical/mathematical intuition on the behaviour of the action.

The behaviour of the superpotential $h(x)$ manifests in the action via the nature of its critical points. Special attention is given to its critical points: we could examine the number of fixed points $\partial h(x_c) = 0$, and determine if the fixed points are maximal or minimal $\partial^2 h(x_c) > 0, \partial^2 h(x_c) < 0$. We would however need to constrain the first derivative of the superpotential to be non-degenerate. The action can then be said to be greatly dependent on the behaviour of the polynomial function. Hence, this formulation of the action hints that we should examine the superpotential and its fixed points in full detail later on. This formulation also provides us with a physical theory to examine the Morse Theory where we associate the superpotential $h(x)$ with the Morse function.

Returning to our discussion in the previous section, the partition function is non-zero and its value is determined by the set of fixed points. We say that the integral localises in the neighbourhood of the set of fixed points. This is the localisation principle: *for a given set of supersymmetric transformations, the partition function localises at loci where the right hand side of the fermionic transformation is zero* [2].

We now expand $h(x)$ about the critical points x_c. Consider $h(x)$ to be a generic polynomial of order n with a maximum of $n-1$ critical points, x_c.

$$h(x) = h(x_c) + \frac{\alpha_c}{2}(x - x_c)^2 + \dots, \tag{2.3.2}$$

we have assumed that $h(x)$ is a continuously differentiable function.

We may relax this condition to consider functions that admit up to the second order derivative, i.e a class C^2 function. We need only keep up to the leading term given that higher-ordered terms brings us significantly away from the critical point when $\partial h(x) \neq 0$ which is irrelevant to our discussion.

Substituting Eq. (2.3.2) into our expression for the partition function in Eq. (2.2.6), we obtain

$$Z = \sum_{x_c} \int \frac{dx\, d\psi_1 d\psi_2}{\sqrt{2\pi}} e^{-\frac{1}{2}\alpha_c^2 (x-x_c)^2 + \alpha_c \psi_1 \psi_2} = \sum_{x_c} \frac{\alpha_c}{|\alpha_c|} = \sum_{x_c} \frac{h''(x_c)}{|h''(x_c)|}. \tag{2.3.3}$$

The expression reduces to an integer and there are only three possible values, $Z = \pm 1$ or 0. We can see this from the reduced expression of Z,

$$Z_{\text{reduced}} = \sum_{x_c : \partial h|_{x_c} = 0} \frac{\partial^2 h(x_c)}{|\partial^2 h(x_c)|}, \tag{2.3.4}$$

which implies that if n is odd, there would be an even number of critical points, hence there would be an equal number of positive $\partial^2 h$ as negative ones (equal number of minimal and maximal points). This may be expressed as follows:

$$\begin{aligned} Z_{\text{reduced}} &= \sum_{x_c : \partial h|_{x_c} = 0,} \frac{\partial^2 h(x_c)}{|\partial^2 h(x_c)|}, \\ &= \sum_{j=1}^{n/2} (-1) + \sum_{i=1}^{n/2} (1) = 0, \end{aligned} \tag{2.3.5}$$

where we summed over the negative terms in the first summation, and the positive terms in the second summation.

If n is even,

$$Z_{\text{reduced}} = \begin{cases} +1, & \text{for} \quad \alpha_c > 0, \\ -1, & \text{for} \quad \alpha_c < 0, \end{cases} \tag{2.3.6}$$

where $\partial^2 h(x_c) = \alpha_c$.

That the partition function gives an integer is intriguing but can be seen from our results in Eq. (2.1.22). The counting is then dependent on the number of maximal and minimal points of the superpotential $h(x)$. Not only that, we also notice that we have effectively reduced the dimensions of the integration (from a continuous integration to a discrete summation). In higher dimensions, this feature will be very useful in evaluating ill-defined path integrals. The answer to what is being counted will become clearer when we discuss a related one-dimensional QFT, i.e. Quantum Mechanics. In fact, the result hints at a further insight.

2.4 Deformation Invariance

We may also notice that because the partition function is expressed up to the sign of $\partial^2 h(x_c)$, we may arbitrarily rescale the function $h(x) \mapsto \lambda h(x)$ for $\lambda > 1$. The rescaling would not change the value of the partition function, given that the leading term in the expansion of $h(x)$ does not change. This would provide justification that semi-classical (first order corrections) approximations are exact.

We will now examine the variation in $h(x)$ of the kind $h \to h + \rho$. Under such a transformation/deformation in the superpotential h(x), the variation of the action is found to be

$$\delta_\rho S \approx \partial h \partial \rho - \partial^2 \rho \psi_1 \psi_2, \tag{2.4.1}$$

keeping $\partial\rho$ up to the first order. We claim that such the deformation of the superpotential leaves the action invariant under supersymmetry.

The proof is two-fold, we first consider $f = \delta g$, where δg denotes the variation of g under some symmetry. We look for correlation functions where δg is a variation under supersymmetry which are then evaluated to be zero.

$$\langle f \rangle = \int \delta g e^{-S} = \int \delta(g e^{-S}) = 0. \tag{2.4.2}$$

More specifically, we consider $g = \partial\rho(x)\psi_1$, where ρ is a small value in the limit of $x \to \pm\infty$ as compared to h. Such a condition imposed on ρ is motivated by the fact that we ask that the boundary terms in Eq. (2.4.2) are insignificant.

It is then found that, for $\epsilon^{(1)} = \epsilon^{(2)} = \epsilon$ and $f = \delta_\epsilon g$

$$\begin{aligned} f = \delta_\epsilon g &= \partial^2 \rho \delta x \psi_1 + \partial \rho(x) \delta \psi_1, \\ &= \epsilon(\partial\rho\partial h - \partial^2 \rho \psi_1 \psi_2). \end{aligned} \tag{2.4.3}$$

And since $\langle \delta_\epsilon g \rangle = 0$,

$$\Big\langle \partial\rho\partial h - \partial^2 \rho \psi_1 \psi_2 \Big\rangle = 0 \tag{2.4.4}$$

It then follows that

$$\langle \delta_\rho S \rangle = 0. \tag{2.4.5}$$

Hence, this implies that the partition function is invariant under such a transformation. Suppose, h is a polynomial of order n, then ρ could be a lower-order polynomial with the vanishing argument still preserved.

Deformation invariance is hence useful in evaluating the partition function, given that we would be able to apply a rescalling $h \to \lambda h$ with $\lambda \gg 1$. As such, the action is very large and the weight e^{-S} would be very small, except in the vicinity of the critical points of h.

Bridging our work on localisation and deformation invariance (of the action under a rescalling and/or translation of the superpotential) we may notice that the number of critical points is reduced under appropriate translation of the superpotential. From our basic understanding of polynomial functions, it is apparent that this means that our result in Eqs. (2.3.5) and (2.3.6) is a topological invariant quantity of the space. This topological invariant quantity may represent some intrinsic physical property (like a quantum number) of our theory in zero-dimensions and perhaps for higher dimensions too.

The result found here may also be extended to n bosonic variables where required. We skip the discussion here to develop our ideas for the one-dimensional theory.

Chapter III

One-Dimensional SUSY QFT

We will now begin our discusssion of a one-dimensional quantum field theory. A one-dimensional theory, or more specifically a (0+1) dimension theory, is one parametrised by time. A one-dimensional curve parametrised by time t (or proper time τ) hence describes a point-like particle tracing out a worldline in Minkowski space. The fields and Lagrangian of our one-dimensional theory would describe the interactions and dynamics of this particle on the target space (spacetime). The fact that the base manifold M does not have spatial dimensions also suggest that there would not be *spin*. This argees with our expectation that this is a theory on Quantum Mechanics in which, histroically, spin was introduced by hand.

Let us consider the following: $(n+1)$-dimensions quantum field theory, a scalar field defines a field value at each point in spacetime. It is hence seen as a mapping of space time to the real numbers: $(t, x_1, \ldots, x_n) \to \phi(t, x_1, \ldots, x_n)$. For a single particle in $(n+1)$ dimensions, its trajectory is specified in space at each instance in time; this can be seen as a mapping of the real numbers into space: $t \to (x_1(t), \ldots, x_n(t))$. Hence we can see that a field in $(0+1)$-dimensions and a particle in $(1+1)$ dimensions are equivalent description [21].

$$\begin{aligned} \phi &: t \to \phi(t), \\ x &: t \to x(t). \end{aligned} \tag{3.0.1}$$

We can observe the parallels between a one-dimensional quantum field theory and quantum mechanics by observing the Lagrangian defined.

3.1 The Path Integral Formalism

It's fairly intuitive that the one-dimensional space on which we formulate the QFT is either a finite interval I, the real line $\mathbb{R}$ or the circle S^1. It can also be seen that there exists homomorphisms between the spaces. Preliminarily, we might expect there to be equivalent results for I and S^1. This might remind us of results from the one-dimensional Ising

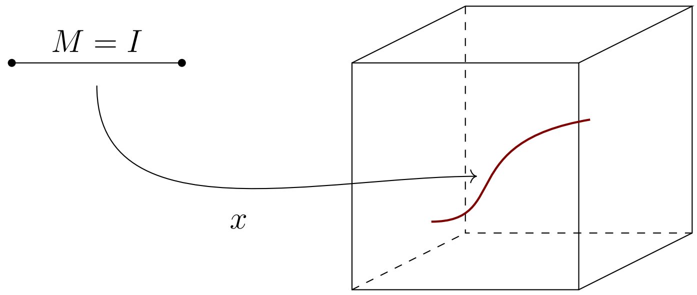


Figure 3.1: Map of $M = I, \mathbb{R}$ or S^1 to (N, g) by the scalar field x; $x : I, \mathbb{R}$ or $S^1 \to (N, g)$.

chain/ring in which the partition functions of the two are equivalent for large numbers, where the bulk behaviour dominates.

Generalising our theory from zero-dimensions, discussed previously in Chapter II, we introduce a single bosonic scalar field ϕ,

$$\phi : I, \mathbb{R} \text{ or } S^1 \to \mathbb{R}, \tag{3.1.1}$$

or more generally, ϕ maps M to some Riemannian manifold (N, g). Suppose there exists a chart $U \subseteq N$ with local coordinates $x^i = (x^1, x^2, \ldots, x^n)$ where $n = \dim(N)$, the scalar field ϕ can be rewritten as

$$x^i \circ \phi = \phi^i. \tag{3.1.2}$$

We will be examining the theory for a simple target space such as the real line first. In the case where $M = I$, we will need to specify boundary conditions on the field in the path integral, i.e. $x(t_1)$ and $x(t_2)$, and this would corresponds to providing start and end points to compute matrix elements.

3.1.1 The Path Integral for One-Dimensional Theory

We will now further consider the one-dimensional generalisation of our partition function

$$Z(x_2, t_2; x_1, t_1) = \int DX(t) e^{iS(X)}. \tag{3.1.3}$$

The partition function here is also otherwise known as the path integral. The integration is evaluated over all possible paths connecting the two points $x(t_1) = x_1$ and $x(t_2) = x_2$. It would then seem natural to impose from both physical and mathematical point of view that the calculations are time ordered – we'll see this more explicitly when we discuss the evaluation of the partition function in Section 3.2.2.

The integration carried out over the space of such maps weighted by the factor of $e^{iS(x)}$ is a sigma model. As seen, the action is defined to be real and hence, the paths are weighted by a complex exponential factor (an oscillatory term). Because of the complex exponential factor, the convergence of the integral is a subtle problem. It's often argued that the contributions of the non-classical paths with non-zero δS, i.e. paths where the

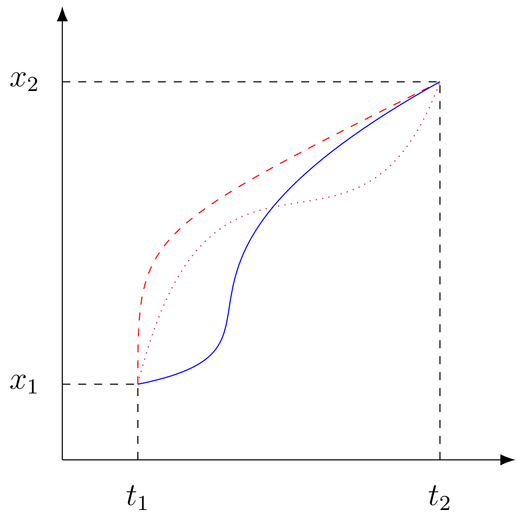


Figure 3.2: A simple representation of the path integral.

action is not stationary, will interfere destructively and contribute to the convergence of the path integral.

Alternatively, we can avoid this problem by considering the "Euclidean theory" by "Euclideanising" the time coordinate t by a Wick rotation: $t \to -i\tau$. Hence arriving at the Euclidean action $S_E(x)$, $S(x) \to iS_E(x)$.

$$Z_E(x_2, t_2; x_1, t_1) = \int_{x(\tau_1)=x_1}^{x(\tau_2)=x_2} Dx(\tau) e^{-S_E(x)}. \tag{3.1.4}$$

We can then interpret the partition function in both Eqs. (3.1.3) and (3.1.4) to be the *probability amplitude* to find a particle 'relocating' from $x_1(t_1)$ to $x_2(t_2)$. We note that as in quantum mechanics, the probability amplitude is a complex number.

Classically, a particle's trajectory is deterministic and determined by it's initial and final boundary conditions. The path integral however gives us further insights that would not have been observed from the canonical quantisation scheme: the trajectory of a particle is only only determined by its boundary conditions but it takes into account of all the possible intermediate steps. This interpretation helps us to carry out computations of quantum mechanical probability amplitudes by considering the Monte Carlo algorithm.

3.2 Supersymmetric Quantum Mechanics

The focus of the supersymmetric quantum mechanics theory we are developing here would be to derive supersymmetric ground states. The existence of supersymmetric ground states determines if supersymmetry is broken at low energies. Simply said, supersymmetry is unbroken if the ground state energy is zero; it is broken if it is larger than zero. We will see how this can be determined in the later sections.

Our ability to evaluate correlation functions would be limited to those that preserve a part of the supersymmetry. This would be done by employing our results from the localisation principle and deformation invariance as discussed in Chapter II.

3.2.1 Single-Variable Potential Theory

In order to make sense of Eq. (3.1.4), let us consider the single-variable potential case for a supersymmetric theory. We generalise our potential theory for zero-dimensions with a single variable x here and introduce its superpartner, ψ, a complex fermion. This is hence the supersymmetric version of our one-dimensional theory in the previous section. We consider the classical supersymmetric Lagrangian to be given by

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\left(h'(x)\right)^2 + \frac{i}{2}\left(\overline{\psi}\dot{\psi} - \dot{\overline{\psi}}\psi\right) - h''(x)\overline{\psi}\psi, \tag{3.2.1}$$

where $\overline{\psi}$ is the complex conjugate of ψ, $\overline{\psi} = \psi^\dagger$.

We may recognise the second term to be equivalent to $-V(x)$ while the last term may be interpreted as a coupling term between the bosonic and fermionic variables. We may treat the fermionic variable and its conjugate as independent variables. The Lagrangian is real as can be seen from $\left(\overline{\psi}\psi\right)^\dagger = \psi^\dagger\overline{\psi}^\dagger = \overline{\psi}\psi$.

We then consider the following set of transformations of the fields x, ψ and $\overline{\psi}$:

$$\begin{aligned}
\delta x &= \epsilon\overline{\psi} - \overline{\epsilon}\psi, \\
\delta\psi &= \epsilon\left(i\dot{x} + h'(x)\right), \\
\delta\overline{\psi} &= \overline{\epsilon}\left(-i\dot{x} + h'(x)\right),
\end{aligned} \tag{3.2.2}$$

where $\epsilon = \epsilon_1 + i\epsilon_2$ is the complex fermionic parameter and $\overline{\epsilon}$ denotes its complex conjugate. As the variational parameter is fermionic, this is a fermionic symmetry. The variation in the Lagrangian can be shown to be a total derivative, $\delta L = \frac{d}{dt}(\ldots)$, and therefore the action is invariant.

The variation of the Lagrangian in terms of arbitrary transformation is

$$\delta L = \dot{x}\delta\dot{x} - h'(x)h''(x)\delta x + \frac{i}{2}\left[\delta\overline{\psi}\dot{\psi} + \overline{\psi}\delta\dot{\psi} - \delta\dot{\overline{\psi}}\psi - \dot{\overline{\psi}}\delta\psi\right], \tag{3.2.3}$$

and expanding in terms of the supersymmetric transformations,

$$\begin{aligned}
\delta L = &\,\dot{x}\left(\epsilon\dot{\overline{\psi}} - \overline{\epsilon}\dot{\psi}\right) - h'(x)h''(x)\left(\epsilon\overline{\psi} - \overline{\psi}\psi\right) \\
&+ \frac{i}{2}\left[\overline{\epsilon}\left(-i\dot{x} + h'(x)\right)\dot{\psi} - \overline{\epsilon}\psi\frac{d}{dt}\left(-i\dot{x} + h'(x)\right)\right] \\
&+ \frac{i}{2}\left[-\dot{\overline{\psi}}\epsilon\left(i\dot{x} + h'(x)\right) - \epsilon\overline{\psi}\frac{d}{dt}\left(i\dot{x} + h'(x)\right)\right] \\
&- h''(x)\left[\overline{\epsilon}\left(-i\dot{x} + h'(x)\right)\psi + \overline{\psi}\epsilon\left(i\dot{x} + h'(x)\right)\right], \\
= &\,\epsilon\frac{d}{dt}\left[\left(\frac{1}{2}\dot{x} + \frac{i}{2}h'(x)\right)\overline{\psi}\right] + \overline{\epsilon}\frac{d}{dt}\left[\left(\frac{-1}{2}\dot{x} + \frac{i}{2}h'(x)\right)\psi\right].
\end{aligned} \tag{3.2.4}$$

We conclude then that the transformation defined is indeed a symmetry of the system. Our classical system with the Lagrangian shown in Eq. (3.2.1) hence has a supersymmetry generated by Eq. (3.2.2).

Following Noether's Theorem, i.e. considering $\epsilon = \epsilon(t)$, we would find the conserved changes to be

$$Q = \overline{\psi}\left(i\dot{x} + h'(x)\right), \qquad \overline{Q} = \psi\left(-i\dot{x} + h'(x)\right). \tag{3.2.5}$$

It is easily seen that $\overline{Q} = Q^{\dagger}$. Hence, we say that there are two real supercharges. We may now attempt to quantise the system. First, we determine the canonical momenta to be

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}} &\equiv p_i = \dot{x}, \\ \frac{\partial L}{\partial \dot{\psi}} &\equiv \Pi_{\psi} = -i\overline{\psi}. \end{aligned} \tag{3.2.6}$$

Note that we have considered the left derivative in our calculations such that $\frac{\partial}{\partial \psi_2}\left(\psi_1\psi_2\right) = -\psi_1$ because the derivative would also anti-commute with the Grassmann variables.

The conjugate momenta for the fermionic variable is obtained via a partial integration which leads us to identify the fermionic part of the Lagrangian to be given by $\int dt \left(i\overline{\psi}\dot{\psi} - h''(x)\overline{\psi}\psi\right)$. We then identify this as the first-order Lagrangian whose configuration space coincides with the Hamiltonian phase space [2, 22]. We note that while ψ and $\overline{\psi}$ are complex conjugates and hence independent variables, the fermionic part of the Lagrangian is necessarily determined by only one initial condition [23], considering time derivatives in both ψ and $\overline{\psi}$ would hence not lead to the correct form of the conjugate momenta and introduce additional boundary conditions. This definition for the conjugate momenta would also agree with the standard Lagrangian for Dirac fields.

Generalising our definition of the Hamiltonian in Eq. (1.4.10) to include fermionic variables and our convention for left derivative, we define H_f and L_f to be the fermionic Hamiltonian and Lagrangian respectively, and it follows that

$$H_f(\psi, \Pi_{\psi}) = \dot{\psi}\Pi_{\psi} - L_f(\psi, \Pi_{\psi}). \tag{3.2.7}$$

Hence, the conserved quantity under time translations

$$H = \frac{1}{2}p^2 + \frac{1}{2}\left(h'(x)\right)^2 + \underbrace{\frac{1}{2}h''(x)\left[\overline{\psi}, \psi\right]}_{H(\psi,\overline{\psi})}, \tag{3.2.8}$$

where the last term in the Hamiltonian is expressed in terms of the commutator by considering a *Weyl ordering* for fermionic variables.

We impose canonical commutation relations of the following and hence quantise the theory:

$$[x, p] = i, \qquad \{\psi, \pi_{\psi}\} = i. \tag{3.2.9}$$

More importantly, we arrive at the following definition

$$\{\psi, \overline{\psi}\} = 1, \tag{3.2.10}$$

where all other (anti-)commutation relations are zero. From Eq. (3.2.10), we may see that this resembles the familiar fermionic ladder operator structure defined in Fock space.

We first define the ground state $|0\rangle$ to be annihilated by ψ:

$$\psi |0\rangle = 0. \tag{3.2.11}$$

The Fock space has an added structure given by the commutation relations

$$[F, \psi] = -\psi, \quad \left[F, \overline{\psi}\right] = \overline{\psi}, \tag{3.2.12}$$

where $F = \overline{\psi}\psi$ is the fermionic number operator.

The fermionic space is hence trivially spanned by

$$\text{span}(H_f) = \{|0\rangle, \overline{\psi}|0\rangle\}, \tag{3.2.13}$$

a two level system given that we only have two fermionic fields (complex conjugates).

The bosonic operators are then defined with the standard representation in quantum mechanics

$$\hat{x}\Psi(x) = x\Psi(x), \quad \hat{p}\Psi(x) = -i\frac{d}{dx}\Psi(x), \tag{3.2.14}$$

where the Hilbert space (for the bosonic variables) $\mathcal{H}^B$ is the space of square-normalisable wave-functions, $L^2(\mathbb{R}, \mathbb{C})$.

The total Hilbert space of states is

$$\begin{aligned} \mathcal{H} &= \mathcal{H}^B \oplus \mathcal{H}^F \\ &= L^2(\mathbb{R}, \mathbb{C}) |0\rangle \oplus L^2(\mathbb{R}, \mathbb{C})\overline{\psi} |0\rangle . \end{aligned} \tag{3.2.15}$$

Recall that in Eq. (3.2.5)

$$Q = \overline{\psi}\left(ip + h'(x)\right), \qquad \overline{Q} = \psi\left(-ip + h'(x)\right), \tag{39}$$

where we have made the substitution for the momentum operator, and similarly promoted these supercharges to operators.

Hence, we have shown that one-dimensional QFT is the theory of quantum mechanics with added fermionic degrees of freedom defined by the fermionic Langragian term in Eq. (3.2.1). It can be shown that

$$[H, Q] = \left[H, \overline{Q}\right] = 0, \tag{3.2.16}$$

and easily seen that,

$$\begin{aligned} Q &: \mathcal{H}^B \mapsto \mathcal{H}^F, \\ \overline{Q} &: \mathcal{H}^F \mapsto \mathcal{H}^B, \end{aligned} \tag{3.2.17}$$

from

$$\begin{aligned} Q\,|0\rangle &= \overline{\psi}\,(ip + h'(x))\,|0\rangle\,, \\ \overline{Q}\,\overline{\psi}\,|0\rangle &= (-ip + h'(x))\,|0\rangle\,, \end{aligned} \tag{3.2.18}$$

where the fermionic number operator F maybe used to check our results for consistency. This reaffirms that the supercharges are indeed conserved quantities in time.

The supersymmetric states are then defined to be states that are invariant under supersymmetry, in that they get mapped from one subspace to the other, as generated by either Q or $\overline{Q}$ according to Eq. (3.2.17). From our definitions of the supercharges in Eq. (3.2.18), it follows that

$$\{Q, \overline{Q}\} = p^2 + (h'(x))^2 + h''(x)\left[\overline{\psi}, \psi\right] = 2H. \tag{3.2.19}$$

One must careful to not mistake this as a classical result; this result cannot be obtained if the commutation relations of Eqs. (3.2.9) and (3.2.10) do not hold, and they do not in the classical theory.

It follows that a state (the supersymmmetric ground state) has zero energy iff it is annihilated by Q and $\overline{Q}$ (from Eq. (3.2.19)):

$$H\,|\alpha\rangle = 0 \iff Q\,|\alpha\rangle = \overline{Q}\,|\alpha\rangle\,. \tag{3.2.20}$$

This is distinctively different from the typical treatment in quantum mechanics that we study in which the ground state energy is non-zero. Also, typically in quantum field theory, the vacuum energy is set to zero from considerations that it is the energy gap that we are mostly interested in (by normal ordering). If such a state exists, i.e. $H\,|\alpha\rangle = 0$, our supersymmetric particle has a ground state with zero energy. The supersymmetric ground state could not be given a non-zero energy by smooth deformations of the Lagrangian.

3.2.2 General Structure of SUSY Quantum Mechanics

Naturally, we may decompose the Hilbert space in terms of the eigenspaces of the Hamiltonian,

$$\mathcal{H} = \bigoplus_{n=0,1,\ldots} \mathcal{H}_{(n)}, \quad H|_{\mathcal{H}_{(n)}} = E_n. \tag{3.2.21}$$

Following the general convention, we see that $E_0 = 0 < E_1 < E_2$, i.e. the energies are non-degenerate and increasing. Given that we have shown the supercharges commute with the Hamiltonian in Eq. (3.2.19), this further suggests that the number of energy levels is preserved. The invariance in the number of energy levels will be important when we calculate the exact supersymmetric ground states using the localisation principle. In addition, the fact that we are able to express the Hamiltonian H in terms of the supercharges should already have been a hint that $Q, \overline{Q}$ commute with H.

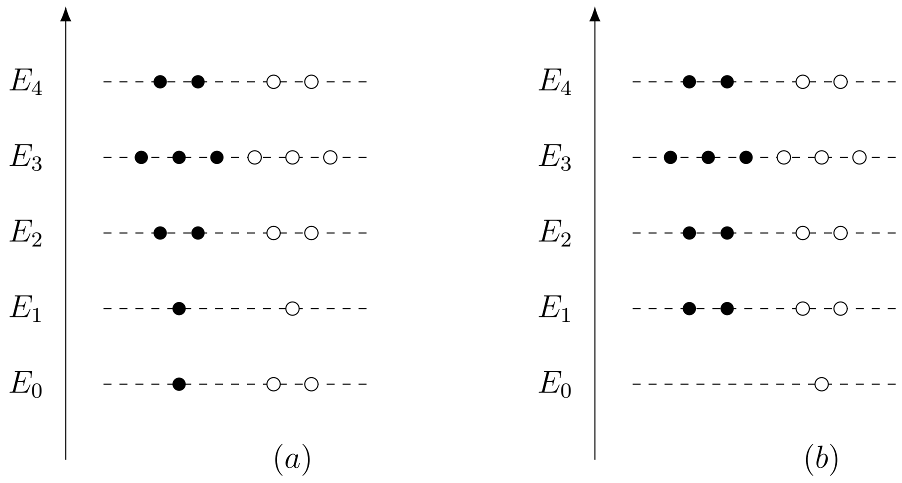


Figure 3.3: The solid circles denote bosonic states while empty circles denote fermionic states. (a): Notice that the excited states are all paired up with the exception of the ground state in which there is an additional fermionic state. We say that the ground state is fermionic. (b): The degeneracy in the ground state is lifted by a continuous deformation. The Witten index of the system is -1. Energy states higher than E_4 are intentionally not shown here.

The formulation of a $\mathbb{Z}_2$ graded structure admits a further decomposition into even and odd subspaces

$$\mathcal{H}_{(n)} = \mathcal{H}^B_{(n)} \oplus \mathcal{H}^F_{(n)}, \tag{3.2.22}$$

and hence naturally, the supercharges map one subspace to the other:

$$Q, \overline{Q} : \mathcal{H}^B_{(n)} \to \mathcal{H}^F_{(n)};\ \mathcal{H}^F_{(n)} \to \mathcal{H}^B_{(n)}. \tag{3.2.23}$$

Given that the number of energy eigenstates is invariant under the supercharges, let us consider the linear combination $Q_1 := Q + \overline{Q}$. From Eq. (3.2.19) we may see that

$$Q_1^2 = 2H. \tag{3.2.24}$$

Notice also that Q and $\overline{Q}$ are individually not Hermitian, while Q_1 is Hermitian. Hence, Eq. (3.2.24) seems to be a more suitable expression for the Hamiltonian.

In addition, we also arrive at the following important result:

$$H = \frac{1}{2}\{Q, \overline{Q}\} \geq 0, \tag{3.2.25}$$

i.e. our Hilbert space is positive semi-definite. Such is the general definition of a theory of supersymmetric quantum mechanics. The theory encompasses a positive definite $\mathbb{Z}_2$-graded Hilbert space of states $\mathcal{H}$ with an even Hamiltonian operator and odd supercharges.

Notice that Q_1 maps $\mathcal{H}^B_{(n)} \to \mathcal{H}^F_{(n)}$ and $\mathcal{H}^F_{(n)} \to \mathcal{H}^B_{(n)}$. It can also be shown that

$$\{Q_1, (-1)^F\} = 0, \tag{3.2.26}$$

and it follows that

$$[H, Q_1] = 0. \tag{3.2.27}$$

It is common to define a theory of supersymmetric quantum mechanics by the following commutation relations:

$$\begin{aligned} Q^2 = Q^{\dagger\,2} &= 0, \\ \{Q, Q^\dagger\} &= 2H, \end{aligned} \tag{3.2.28}$$

as in [1, 24, 25] and other similar papers on this topic. Hence, we see that it follows from our discussion in the previous section that $\overline{Q}$ and $Q^\dagger$ may be used interchangeably. We will opt to use $Q^\dagger$ in our following discussion.

Notice that Q_1 preserves each energy level $[Q_1, H] = 0$, mapping $\mathcal{H}^B_{(n)}$ to $\mathcal{H}^F_{(n)}$ and vice versa. For $E_n > 0$ at the nth level, Q_1 is invertible and defines an isomorphism

$$\mathcal{H}^B_{(n)} \cong \mathcal{H}^F_{(n)}. \tag{3.2.29}$$

This follows from our discussion that the number of energy levels is invariant under the supercharges i.e. an isomorphism given by a linear combination of the supercharges exists. Hence, there exists a pairing between bosonic and fermionic excited states for $n \geq 1$. This however does not include the case for $n = 0$ since the transformation is not invertible for $n = 0$, hence such a pairing may not necessarily exists (see Fig. 3.3).

From our results in Section 2.4, it can be shown that a continuous deformation of the theory preserves the supersymmetry and hence what this suggests is that the difference in the number of (exact) bosonic ground states and (exact) fermionic ground states is an invariant quantity. This is defined to be the Witten index

$$\dim \mathcal{H}^B_{(0)} - \dim \mathcal{H}^F_{(0)} = \mathrm{tr}\left\{(-1)^F e^{-\beta H}\right\}. \tag{3.2.30}$$

The operator $(-1)^F$ inserted into the trace is the parity operator, as defined in Eqs. (2.1.10) and (2.1.11) of Section 2.1.1. Notice that this parity operator operator has a similar definition to that of found in standard quantum mechanics/field theory. For a more physical idea, the insertion of the parity operator allows for the cancellation of the non-zero energy excited bosonic and fermionic states; given that there is a pairing in the excited states, the parity operator introduces a negative sign for the fermionic ones hence allowing for the cancellation to occur. Our interests is in determining the supersymmetric ground states and hence it seems natural that we would want to remove the contribution from the excited states.

The independence of the contribution from the excited states also further strengthens our method of calculation in that it is resistant to ultraviolet divergences [25]. This is because the calculation of the Witten index involves only the low-lying states. Hence, so long as the pairing between bosonic and fermionic states exists even in the ultraviolet limit, i.e. supersymmetry holds in the limit, the cancellations due to the parity operator would still hold in our calculation of the Witten index [1].

This invariant quantity may be associated as a topological invariant quantity defined on the target space, known as the supersymmetric index or the Witten index, introduced in [1, 25]. It may also be denoted simply as $\operatorname{tr}(-1)^F$. It can be further shown that the index does not depend on β and hence reduces to the simplified expression[1]. This will be shown later in Section 3.2.4.

We now claim that the study of the symmetric ground states may be described by the cohomology group of target space. From Eq. (3.2.28), $Q^2 = 0$ suggests that we have $\mathbb{Z}_2$-graded complex of vector spaces

$$\mathcal{H}^F \xrightarrow{Q} \mathcal{H}^B \xrightarrow{Q} \mathcal{H}^F \xrightarrow{Q} \mathcal{H}^B, \tag{3.2.31}$$

or

$$\Lambda^1(V) \xrightarrow{Q} \Lambda^0(V) \xrightarrow{Q} \Lambda^1(V) \xrightarrow{Q} \Lambda^0(V), \tag{3.2.32}$$

following our previous discussion of Grassmann numbers in Section 2.1.

Hence, the cohomology structure of this complex generated by the supercharges may then be defined to be

$$\begin{aligned} H^B(Q) &:= \frac{\ker(Q) : \mathcal{H}^B \to \mathcal{H}^F}{\operatorname{im}(Q) : \mathcal{H}^F \to \mathcal{H}^B}, \\ H^F(Q) &:= \frac{\ker(Q) : \mathcal{H}^F \to \mathcal{H}^B}{\operatorname{im}(Q) : \mathcal{H}^B \to \mathcal{H}^F}, \end{aligned} \tag{3.2.33}$$

where the left-hand side defines the corresponding cohomology group in the bosonic or fermionic states generated by Q.

We may then also define a state $|n\rangle$ with the property that for $H|n\rangle = E_n|n\rangle$ and $Q|n\rangle = 0$ to be Q-closed. Such states hence belong the $\ker(Q) \subseteq \mathcal{H}^B$. Yet, state $|n\rangle$ is also Q-exact given that

$$1 = \frac{\left(QQ^\dagger + Q^\dagger Q\right)}{(2E_n)} \tag{3.2.34}$$

on $\mathcal{H}_{(n)}$, and hence

$$|n\rangle = Q\left(\frac{Q^\dagger|n\rangle}{(2E_n)}\right). \tag{3.2.35}$$

This is a result that similarly can be arrived at from the Poincaré lemma that states that the exact forms $B^n := \operatorname{im} Q \subseteq \Lambda^n(V)$ and closed forms $Z^n := \ker(Q) \subseteq \Lambda^n(V)$ are equal if the manifold $M = \mathbb{R}^{\dim m}$ for $n > 0$,

$$B^n = Z^n. \tag{3.2.36}$$

n is denoted here as the parity of the space (even or odd) of the $\mathbb{Z}_2$ graded structure.

[1] The convergence of the index due to the infinite summation of all the states in the Hilbert is addressed in [25].

It may be generalised to consider that for a given coordinate patch U which is locally flat, a closed r-form on U is also exact [4]. The fact that we found that supersymmetric ground states are both closed and exact suggests that they are isolated points on the manifold M.

Hence, from the result above, the cohomology groups of Q are purely the supersymmetric ground states

$$H^B(Q) = \mathcal{H}^B_{(0)}, \quad H^F(Q) = \mathcal{H}^F_{(0)}. \tag{3.2.37}$$

What this also demonstrates is that the study of supersymmetric ground states is faciliated by the cohomology of the Q-operator. Mathematically, this also hints at an association between the supercharges and the exterior derivative and its adjoint. Seemingly, the natural language to understand the supercharges would be to introduce exterior derivatives, which follows given the association defined originally of Grassmann algebra and the exterior algebra.

Our results here may be extended for a finer grading such as a $\mathbb{Z}$-grading. This is the case if the Hermitian operator F takes on integral eigenvalues such that $e^{\pi i F} = (-1)^F$. Physically, this means that we have a richer structure that allows for a greater number of fermions, the Hilbert space is also enlarged. This will be the case for the multivariable scenario. Generalising our results from Eqs. (3.2.33) and (3.2.37) we have:

$$\mathcal{H}^B_{(0)} = \bigoplus_{p \text{ even}} H^p(Q), \quad \mathcal{H}^F_{(0)} = \bigoplus_{p \text{ odd}} H^p(Q), \tag{3.2.38}$$

and

$$\mathrm{tr}(-1)^F = \sum_{p\in\mathbb{Z}} (-1)^p \dim H^p(Q) = \chi(M), \tag{3.2.39}$$

whch equates the Witten index to the Euler characteristic of the complex. Hence, the physical properties of the system is translated to a topological invariant, coherent to our definition for the Witten index.

The one-dimensional partition function for our supersymmetric theory and the Witten index can hence be defined in the path-integral form on a circle of circumference β as

$$\begin{aligned} Z(\beta) = \mathrm{tr}\left\{e^{-\beta H}\right\} &= \int \mathcal{D}X\mathcal{D}\psi\mathcal{D}\overline{\psi}|_{\mathrm{AP}} e^{-S(X,\psi,\overline{\psi})}, \\ \mathrm{tr}\left\{(-1)^F\right\} = \mathrm{tr}\left\{(-1)^F e^{-\beta H}\right\} &= \int \mathcal{D}X\mathcal{D}\psi\mathcal{D}\overline{\psi}|_{\mathrm{P}} e^{-S(X,\psi,\overline{\psi})}. \end{aligned} \tag{3.2.40}$$

where the subscript AP and P refers to the anti-periodic and periodic boundary conditions imposed on the fermionic fields.

In order to see why the initial boundary condition is anti-periodic in the fermionic fields before inserting the parity operator $(-1)^F$, we would need to consider a time-ordered product of fermionic variables $\mathcal{T}\left\{\overline{\psi}(t_1)\psi(t')\right\}$ defined on the time interval $[0, \beta]$.

The initial conditions define $t' < t_1 < \beta$. Time-ordering is defined by the following operation:

$$\mathcal{T}\left\{\overline{\psi}(t_1)\psi(t')\right\} := \left\{ \begin{array}{ll} \overline{\psi}(t_1)\psi(t), & \text{for } t' < t_1 < \beta \\ -\psi(t')\overline{\psi}(t_1), & \text{for } t_1 < t' \leq \beta \end{array} \right\} \tag{3.2.41}$$

Hence, as t increases from $t < t_1$ to $t_1 < t \leq \beta$, a time-ordered fermionic pair (product) would gain a negative sign. Hence, it follows that the insertion of the parity operator, acting on the fermionic states before evaluation of the path integral, would give rise to a periodic boundary.

While we have considered this result for a path integral defined on a circle of circumference β, this result can be shown to apply for the real line $\mathbb{R}$. This can be seen from the analysis done in Chapter 10.1 of [2] for sigma models on the circle S^1 and real line $\mathbb{R}$.

We can observe that the Witten index is in fact independent of the circumference of the circle. The change in the circumference of the circle is equivalent to the insertion of H in the path-integral

$$\int \mathcal{D}X\mathcal{D}\psi\mathcal{D}\overline{\psi}|_{\mathrm{P}} He^{-S(X,\psi,\overline{\psi})}. \tag{3.2.42}$$

However,

$$\int \mathcal{D}X\mathcal{D}\psi\mathcal{D}\overline{\psi}|_{\mathrm{P}} He^{-S(X,\psi,\overline{\psi})} = \int \mathcal{D}X\mathcal{D}\psi\mathcal{D}\overline{\psi}|_{\mathrm{P}} \frac{1}{2}\{Q, Q^\dagger\}e^{-S(X,\psi,\overline{\psi})} = 0, \tag{3.2.43}$$

given that $QQ^\dagger$ (and conversely $Q^\dagger Q$) can be considered a variantion of the field $Q^\dagger$ under supersymmetric variations.

Furthermore, as shown in Section 2.4, if the correlation function of fields can be expressed as variations of supersymmetry, the partition function turns out to be zero. This would similarly apply to the insertion of H into the Witten index. Hence, we conclude that it is β independent.

3.2.3 Determination of SUSY Ground States

Let us determine the general form of the supersymmetric ground states for an arbitrary superpotential $h(x)$. In Section 3.2.1 we determined that the fermionic part of the Hilbert space is a two-dimensional space spanned by

$$|0\rangle, \overline{\psi}|0\rangle. \tag{3.2.13 revisited}$$

Given also that we defined the state $|0\rangle$ to be annihilated by the “lowering operator” as in Eq. (3.2.11), the fermionic operators have matrix representation of the form

$$\psi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \overline{\psi} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.2.44}$$

Respectively, our supercharges would then have matrix representation of the following

$$Q = \overline{\psi}\left(ip + h'(x)\right) = \begin{pmatrix} 0 & 0 \\ \frac{d}{dx} + h'(x) & 0 \end{pmatrix}, \quad Q^\dagger = \psi\left(-ip + h'(x)\right) = \begin{pmatrix} 0 & -\frac{d}{dx} + h'(x) \\ 0 & 0 \end{pmatrix}. \tag{3.2.45}$$

The supersymmetric ground states Ψ can then be expressed as a linear superposition of the terms that span the Hilbert space such that

$$\begin{aligned}\Psi &= f_1(x)\,|0\rangle + f_2(x)\overline{\psi}\,|0\rangle\,,\\ Q\Psi &= Q^\dagger\Psi = 0.\end{aligned} \tag{3.2.46}$$

Solving for $f_1(x)$ and $f_2(x)$, gives the linear first order differential equations

$$\begin{aligned}\left(\frac{d}{dx} + h'(x)\right) f_1(x) = 0,\\ \left(-\frac{d}{dx} + h'(x)\right) f_2(x) = 0,\end{aligned} \tag{3.2.47}$$

with well known solutions

$$f_1(x) = c_1 e^{-h(x)}, \quad f_2(x) = c_2 e^{h(x)}. \tag{3.2.48}$$

Imposing normalisable solutions for the bosonic part would lead us to consider the behaviour of the superpotential as $x \to \pm\infty$. It is clear that the $f_1(x)$ and $f_2(x)$ are not normalisable in the same regime. Instead, there are three possible cases to consider.

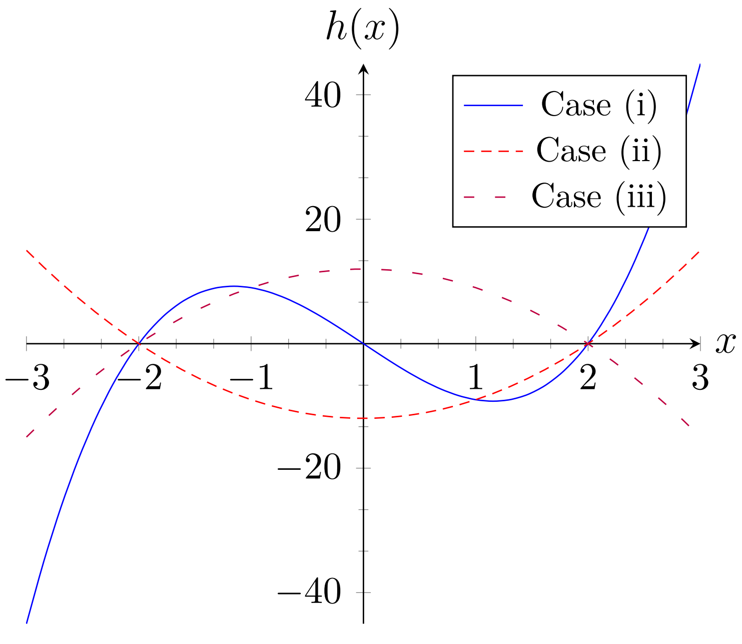


Figure 3.4: Examples for possible forms of the superpotential, $h(x)$. Case (i): $\lim_{x\to\pm\infty} h(x)$ tends to $\pm\infty$; Case (ii): $\lim_{x\to\pm\infty} h(x)$ tends to $+\infty$; Case (iii): $\lim_{x\to\pm\infty} h(x)$ tends to $-\infty$.

For the case of a cubic expression of $h(x)$ as shown in Fig. 3.4 above, in the limit that $x \to \infty$, $f_2(x)$ would be non-normalisable, while in the limit of $x \to -\infty$, $f_1(x)$ is non-normalisable. Hence, given an odd polynomial function for the superpotential, there are no supersymmetric ground states,

$$\operatorname{tr}(-1)^F = 0. \tag{3.2.49}$$

Note that the $h(x)$ is defined to be a non-degenerate polynomial function of x.

From a similar analysis, we would find that in Case (ii), $f_1(x)$ is normalisable while in Case (iii), $f_2(x)$ is normalisable. These two cases in which the superpotential is an even polynomial function are of greater interest: Case (ii) would suggest that the supersymmetric ground state is bosonic, while Case (iii) suggests that it is fermionic. This result can be generalised for a multivariable case in which each field variable would give rise to a different supersymmetric ground state.

$$\Psi = \begin{cases} e^{-h(x)} \left|0\right\rangle, & \text{for Case (ii)}, \\ e^{h(x)}\overline{\psi} \left|0\right\rangle, & \text{for Case (iii)}. \end{cases} \tag{3.2.50}$$

Correspondingly,

$$\operatorname{tr}(-1)^F = \begin{cases} 1, & \text{for Case (ii)}, \\ -1, & \text{for Case (iii)}, \end{cases} \tag{3.2.51}$$

the supersymmetric ground state is in $\mathcal{H}^B$ and $\mathcal{H}^F$ respectively. The results here should remind you of our discussion in Section 2.3, more specifically in the cases laid out in Eqs. (2.3.5) and (2.3.6).

An example could be made to check our understanding; for the interested reader, please refer to Appendix B.

3.2.4 Witten Index: The Path Integral Approach

Recall that in Section 3.2.2 we found the following expressions for the path integral and Witten index

$$\begin{aligned} Z(\beta) = \operatorname{tr}\left\{e^{-\beta H}\right\} &= \int \mathcal{D}x\mathcal{D}\psi\mathcal{D}\overline{\psi}|_{\mathrm{AP}}e^{-S_E(x,\psi,\overline{\psi})}, \\ \operatorname{tr}\left\{(-1)^F\right\} = \operatorname{tr}\left\{(-1)^F e^{-\beta H}\right\} &= \int \mathcal{D}x\mathcal{D}\psi\mathcal{D}\overline{\psi}|_{\mathrm{P}}e^{-S_E(x,\psi,\overline{\psi})}, \end{aligned} \tag{3.2.40}$$

with the (effective) Euclidean action given by

$$S_E = \int_0^{2\pi} d\tau \left\{ \frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + \frac{1}{2}\left(h'(x)\right)^2 + \overline{\psi}\frac{d}{d\tau}\psi + h''(x)\overline{\psi}\psi \right\}. \tag{3.2.52}$$

One can easily show by partial integration that the Lagrangian in the given integrand is equivalent to Eq. (3.2.1). The supersymmetric transformations in Eq. (3.2.2) is modified to give

$$\begin{aligned} \delta x &= \epsilon\overline{\psi} - \overline{\epsilon}\psi, \\ \delta\psi &= \epsilon\left(-\frac{dx}{d\tau} + h'(x)\right), \\ \delta\overline{\psi} &= \overline{\epsilon}\left(\frac{dx}{d\tau} + h'(x)\right). \end{aligned} \tag{3.2.53}$$

The localisation principle tells us that the path integral would localise to regions where the supersymmetric transformation of the fermionic fields vanish. This suggests that the result would localise at

$$\frac{dx}{d\tau} = h'(x) = 0, \tag{3.2.54}$$

mapping the action to the set of critical points $x_i \in \{x_1, \ldots, x_N\}$ where N is the maximum number of critical points in $h(x)$.

Solving the differential equation in the integrand by means of linear algebra, we arrive at

$$S_E^{(i)} = \int_0^{2\pi} \left\{ \frac{1}{2}\xi \left(-\frac{d^2}{d\tau^2} + h''(x_i)^2 \right) \xi + \overline{\psi} \left(\frac{d}{d\tau} + h''(x_i) \right) \psi \right\} d\tau, \tag{3.2.55}$$

where $\xi := x - x_i$. Eq. (3.2.55) is the action about critical point x_i given up to the quadratic term. The effective path-integral for the given action is hence

$$\int \mathcal{D}X\mathcal{D}\psi\mathcal{D}\overline{\psi}|_{\mathrm{P}} e^{-S_E^{(i)}} = \frac{\det\left(\partial_\tau + h''(x_i)\right)}{\sqrt{\det\left(-\partial_\tau^2 + h''(x_i)^2\right)}}, \tag{3.2.56}$$

$$= \frac{\Pi_{n\in\mathbb{Z}}\left(in + h''(x_i)\right)}{\sqrt{\Pi_{n\in\mathbb{Z}}\left(n^2 + h''(x_i)^2\right)}} \tag{3.2.57}$$

$$= \frac{h''(x_i)}{|h''(x_i)|}, \tag{3.2.58}$$

where in the last step, the path-integral reduces to an integer, as expected, owing to the cancelation between fermionic and bosonic excited modes. The total Witten index is hence the discrete sum of the path-integral expanded about each critical point x_i,

$$\operatorname{tr}(-1)^F e^{-\beta H} = \sum_{i=1}^{N} \operatorname{sgn}\left(h''(x_i)\right). \tag{3.2.59}$$

More accurately, only the time-independent constant modes contribute to the path-integral. The periodic boundary conditions imposed on the fermionic and bosonic variables is a necessary move such that the supersymmetric transformations listed in Eq. (3.2.53) holds. Without which, we would not be able to impose the localisation principle; recall that localisation occurs in regions where $\delta\psi = 0, \delta\overline{\psi} = 0$. If the periodic boundary conditions do not hold, we would have two distinct conditions that may not hold simultaneously: $-\frac{dx}{d\tau} + h'(x) = 0$, and $\frac{dx}{d\tau} + h'(x) = 0$, which would require that $-\frac{dx}{d\tau} = \frac{dx}{d\tau}$ for periodic boundaries.

It follows that the computation of the partition function is non-trivial. We will however not discuss the computation for the partition function here[2].

In a simple example of a supersymmetric harmonic oscillator as in Appendix B, it can be shown that the Witten index is non-zero thus demonstrating that for this very simple example supersymmmetry is not spontaneously broken. In [24], Witten found that the index was found to be non-zero for a number of interesting cases, particularly in four-dimensions supersymmetric ϕ^4 theory and supersymmetric non-Abelian gauge theories.

[2]For completeness, one may refer to Section 1.5.10 of [4] for information on evaluating the result.

3.3 Non-Linear Sigma Models

We will now consider the supersymmetric theory on a non-trivial target manifold: consider a Riemannian manifold (M, g) as the target manifold with the metric tensor $g_{ij}(x)$. The metric takes two tangent vectors as inputs and gives a real number $g_p(\nu, \mu)$. It provides a basis from measuring distances (lengths, area and volume) on the manifold.

Theories formulated on the basis of scalar maps from a base manifold onto a Riemannian target manifold are known as non-linear sigma models. It is here that we can see the relationship between the topology of the target manifold and the structure of our supersymmetric sigma model. The superpotentials would also take on the interpretation of Morse functions defined on the manifold. We would hence be able to see a realisation of the Morse theory (a mathematical theory) in our supersymmetric model when we determine the supersymmetric ground states.

3.3.1 Classical Theory

Let us also assume that the Riemannian manifold is oriented and compact. Denote a generic set of local coordinates of M by $x^i = x^1, \ldots, x^n$ where $\dim M = n$. Compactness can be relaxed but it helps to look for solutions on which the Riemannian manifold is compact; a compact Riemann manifold M would admit a finite number of critical points $\partial h(x_i) = 0$

Our base manifold $\mathcal{T}$ is parametrised by time t; let us define a bosonic field ϕ, where $\phi : \mathcal{T} \to M$. The bosonic scalar field is represented locally as $x^i \circ \phi = \phi^i$ where the ϕ^i denotes the ith component of the column vector ϕ.

The fermionic variables ψ and $\overline{\psi}$ respectively correspond to sections on $\mathcal{T}$. Formally, they are defined as

$$\psi, \overline{\psi} \in \Gamma\left(\mathcal{T}, \phi^* TM \otimes \mathbb{C}\right), \tag{3.3.1}$$

or smooth maps from $\mathcal{T} \to \phi^* TM \otimes \mathbb{C}$, where $\phi^* TM \otimes \mathbb{C}$ denotes the direct product of the pullback tangent bundle with $\mathbb{C}$[3]. The pullback of TM by ϕ is a bundle $\phi^* TM$ over $\mathcal{T}$, i.e. ψ and $\overline{\psi}$ are tangent vectors on $\mathcal{T}$. ψ is locally represented by $\psi = \psi^i \left(\frac{\partial}{\partial x^i}\right)\Big|_\phi$.

Let us consider a generalisation of the Lagrangian[4] in Eq. (3.2.1) given by

$$\begin{aligned} L = \frac{1}{2} g_{ij} \dot{\phi}^i \dot{\phi}^j + \frac{i}{2} g_{ij} \left(\overline{\psi}^i \nabla_t \psi^j - \nabla_t \overline{\psi}^i \psi^j\right) - \frac{1}{8} R_{ijkl} \psi^i \overline{\psi}^j \psi^k \overline{\psi}^l \\ - \frac{1}{2} g^{ij} \partial_i h \partial_j h - \nabla_i \partial_j h \overline{\psi}^i \psi^j, \end{aligned} \tag{3.3.2}$$

[3] The direct product with $\mathbb{C}$ is introduced given that our fermionic fields are complex variables.

[4] We have chosen the Lagrangian in favour of the representation in [1, 25] over [2]. The results would not differ, nonetheless.

$$
\begin{tikzcd}
\phi^* TM \arrow[r, "\phi_*"] & TM \arrow[d, "\pi"] \\
\mathcal{T} \arrow[u, "\Gamma"] \arrow[r, "\phi"'] & M
\end{tikzcd}
$$

Figure 3.5: ϕ_* represents the pushforward from the pullback tangent bundle ϕ^*TM. While π denotes the projection from the tangent bundle TM to M.

where

$$
\begin{aligned}
\nabla_t \psi^i &= \partial_t \psi^i + \Gamma^i_{jk} \partial_t \phi^j \psi^k, \\
\nabla_i \partial_j h &= \partial_i \partial_j h - \Gamma^k_{ij} \partial_k h.
\end{aligned}
\tag{3.3.3}
$$

define the covariant derivative of ψ^i.

The definition of the Lagrangian agrees with our definition that $\psi, \overline{\psi}$ denotes sections over our target manifold M. Γ^i_{jk} is the Christoffel symbol of the Levi-Civita connection on the Riemannian manifold and $h(x)$ and R_{ijkl} is the Riemann curvature tensor defined for the Riemannian manifold. The covariant derivative ∇_t is defined by considering the pullback of the Levi-Civita connection on M. ψ and $\overline{\psi}$ being odd variables are not defined on manifold M (they are sections of the tangent spaces), hence that would explain why a pullback is required. The 'non-linearity' of our theory is found in the equations of motion of the Lagrangian; more specifically, the solutions to the Euler-Lagrange equations are the geodesic equations which are non-linear.

The Lagrangian is a deformed case in which an arbitrary potential term $h(x)$ on M (the superpotential from our previous cases) is introduced to lift the degeneracy in the supersymmetric ground states. It is very common that the classical vacuum state is degenerate and by enforcing the deformation invariance of the theory we can lift the degeneracy.

We may also identify the supersymmetric transformations for our model to be:

$$
\begin{aligned}
\delta \phi^i &= \epsilon \overline{\psi}^i - \overline{\epsilon} \psi^i, \\
\delta \psi^i &= \epsilon \left(i \dot{\phi}^i - \Gamma^i_{jk} \overline{\psi}^j \psi^k + g^{ij} \partial_j h \right), \\
\delta \overline{\psi}^i &= \overline{\epsilon} \left(-i \dot{\phi}^i - \Gamma^i_{jk} \overline{\psi}^j \psi^k + g^{ij} \partial_j h \right).
\end{aligned}
\tag{3.3.4}
$$

The set of transformations listed in above is found in [2]. If our formulation in the earlier section of Section 3.2.1 is any indication of the generalisation results, we would perhaps interpret the four-fermionic as well as the additional potential term, and $\nabla_i \partial_j h \overline{\psi}^i \psi^j$, in Eq. (3.3.2) to be the fermionic Hamiltonian of the system.

By extension, we would expect the Hamiltonian of the system to be given by

$$H = \frac{1}{2}g_{ij}\phi^i\phi^j + \frac{1}{2}g^{ij}\partial_i h\partial_j h + \frac{1}{8}R_{ijkl}\psi^i\overline{\psi}^j\psi^k\overline{\psi}^l + \nabla_i\partial_j h\overline{\psi}^i\psi^j. \tag{3.3.5}$$

The conjugate momenta of our scalar fields and fermionic fields are respectively[5]:

$$\frac{\partial L}{\partial\dot{\phi}^m} = g_{mj}\dot{\phi}^j + \frac{i}{2}g_{ij}\Gamma^j_{ml}\left(\overline{\psi}^i\psi^l - \overline{\psi}^l\psi^i\right) =: p_m, \tag{3.3.6}$$

$$p_m = g_{mj}\dot{\phi}^j + \frac{i}{2}\left(\Gamma_{i,ml} - \Gamma_{l,mi}\right)\overline{\psi}^i\psi^l \tag{3.3.7}$$

$$= g_{mj}\dot{\phi}^j + i\partial_l g_{im}\left(\overline{\psi}^i\psi^l - \overline{\psi}^l\psi^i\right) \tag{3.3.8}$$

$$\frac{\partial L}{\partial\dot{\psi}^m} = -ig_{mj}\overline{\psi}^j =: \Pi_\psi. \tag{3.3.9}$$

where we use the shorthand notation, $\Gamma_{i,ml} = g_{ij}\Gamma^j_{ml}$.

Notice that the variation of the bosonic fields ϕ^i are along the tangent vectors defined by $\psi, \overline{\psi}$. Since both the infinitesimal variation parameter ϵ ($\overline{\epsilon}$) and fermionic variables ψ ($\overline{\psi}$) are both necessarily sections of $\phi^*TM \otimes \mathbb{C}$, the product terms e.g. $\epsilon\overline{\psi}^i$ are real.

Considering ϵ and $\overline{\epsilon}$ to be time dependent as in Noether's procedure discussed in Section 3.2.1, we arrive at the following supercharges:

$$\begin{aligned}\mathcal{Q} &= \overline{\psi}^i\left(ip_i + \partial_i h\right),\\ \overline{\mathcal{Q}} &= \psi^i\left(-ip_i + \partial_i h\right),\end{aligned} \tag{3.3.10}$$

where p_i is defined the same expression as in Eq. (3.3.8).

There exists a $U(1)$ symmetry in the fermionic fields given by

$$\psi^i \to e^{-i\gamma}\psi^i, \quad \overline{\psi}^i \to e^{i\gamma}\overline{\psi}^i, \tag{3.3.11}$$

with the corresponding Noether's charge, we define the fermionic charge,

$$F = g_{ij}\overline{\psi}^i\psi^j, \tag{3.3.12}$$

which when quantised becomes the fermion number operator of the non-linear sigma model.

Before we continue with our discussion, let us define Q and $\overline{Q}$ to be the supercharges for the case $h(x) = 0$, i.e. in the absence of the additional potential term.

$$\begin{aligned}Q &= \overline{\psi}^i ip_i,\\ \overline{Q} &= -\psi^i ip_i.\end{aligned} \tag{3.3.13}$$

[5]The expression of the conjugate momenta to the bosonic scalar fields differs from that found in [2], it seems that there's a mistake in the text.

3.3.2 Quantisation of Non-Linear Sigma Model

The canonical quantisation of the non-linear sigma model calls for imposing the (anti-)commutation relations (suitable for a $\mathbb{Z}_2$ graded algebra):

$$\left[\phi^i, p_j\right] = \delta^i_j \tag{3.3.14}$$

$$\{\psi^i, \overline{\psi}^j\} = g^{ij}. \tag{3.3.15}$$

Recalling from our discusson in Section 3.2.2, the Hamiltonian of our supersymmetric model may be defined and constrained by the supersymmetric relation

$$\{\mathcal{Q}, \mathcal{Q}^\dagger\} = 2H, \tag{3.3.16}$$

where $\mathcal{Q}$ and $\overline{\mathcal{Q}} = \mathcal{Q}^\dagger$ are as defined in Eq. (3.3.10).

They are further constrained by noting that

$$[F, \mathcal{Q}] = \mathcal{Q}, \quad \left[F, \overline{\mathcal{Q}}\right] = -\overline{\mathcal{Q}}, \tag{3.3.17}$$

which also suggests that the fermion charge operator commutes with the Hamiltonian.

$$[H, F] = 0. \tag{3.3.18}$$

The structure of supersymmetry quantum mechanics on the Riemannian manifold as given by Eq. (3.3.16) and $\mathcal{Q}^2 = \mathcal{Q}^{\dagger\,2} = 0$ further suggests a natural representation for our quantum theory. We may hence be inclined to specify the Hilbert space for our quantum theory as the space of differential forms[6],

$$\mathcal{H} = \Omega(M) \otimes \mathbb{C}, \tag{3.3.19}$$

equipped with a well defined Hermitian inner product

$$(\omega_1, \omega_2) = \int_M \overline{\omega}_1 \wedge \star\omega_2. \tag{3.3.20}$$

In [3], it is further argued that the natural choice of representing the states in terms of differential forms follows from the following illustration[7]: consider a state $|0\rangle$ annihilated by all ψ^i (or $\mathcal{Q}^\dagger$), such a state may retain arbitrary dependence on the bosonic variables x^i. Hence, a state of minimum possible fermion charge is $f(x)\,|0\rangle$, where $f(x)$ is some arbitrary function on M. A state whose fermion charge is greater by n is then $\sum_{i_1\ldots i_n} f_{i_1\ldots i_n}(x)\overline{\psi}^{i_1}\overline{\psi}^{i_2}\ldots\overline{\psi}^{i_n}\,|0\rangle$. Naturally, in the language of differential forms, this correspond to the n-form on M,

$$\sum_{i_1\ldots i_n} f_{i_1\ldots i_n}(x) dx^{i_1} dx^{i_2} \ldots dx^{i_n}. \tag{3.3.21}$$

[6] We drop the tilde on the differential forms here in this section.

[7] In [3] the definition for the fermionic raising and lowering operator is the reverse from the discussion here: It is considered in [3] that ψ gives $F = 1$, while $\overline{\psi}$ gives $F = -1$. We have tweaked the argument presented to follow our convention.

It then follows naturally that the quantised operators ϕ^i, p_i, ψ^i and $\Pi_{\psi,i}$ have the following representations:

$$\begin{aligned}
\phi^i &= x^i \times, \\
p_i &= -i\nabla_i, \\
\overline{\psi}^i &= dx^i \wedge, \\
\psi^i &= g^{ij} i_{\partial/\partial x^j},
\end{aligned} \tag{3.3.22}$$

where i_V is the operation of contraction of the differential form with the vector field V, and ∇_i is the covariant derivative with respective to the local coordinates x^i on M[8].

Following the usual convention, if we define the state $|0\rangle$ to be the state with the lowest fermion number, annihilated by all ψ^i, we would find that there exists a one-to-one correspondence between the states in the fermionic Hilbert space H_f and the space of differential forms $\Omega^n(M)$ where $\dim M = n$. This relationship can be seen from

$$\begin{aligned}
|0\rangle &\leftrightarrow 1 \\
\overline{\psi}^i |0\rangle &\leftrightarrow dx^i \\
\overline{\psi}^i \overline{\psi}^j |0\rangle &\leftrightarrow dx^i \wedge dx^j \\
&\vdots \\
\overline{\psi}^i \overline{\psi}^j \ldots \overline{\psi}^n &\leftrightarrow dx^i \wedge dx^j \wedge \ldots \wedge dx^n,
\end{aligned} \tag{3.3.23}$$

where $dx^i \wedge dx^j \wedge \ldots \wedge dx^n$ denotes the top-form, or the state of maximum fermion number; $\mathcal{Q}$ maps such a state to the the null set $\{0\}$. Naturally, our supercharges are also promoted to operators and have the following representation in terms of the exterior derivative:

$$\mathcal{Q} = d + dx^i \wedge \partial_i h = d + dh\wedge = e^{-h} d e^h =: d_h, \tag{3.3.24}$$

$$\mathcal{Q}^\dagger = (d + dh\wedge)^\dagger = e^h d^\dagger e^{-h} = d_h^\dagger. \tag{3.3.25}$$

Given that $\mathcal{Q}^\dagger$ is defined to be the Hermitian conjugate of $\mathcal{Q}$, it follows naturally that it is the adjoint exterior derivative. Having defined the supercharges, the Hamiltonian is hence automatically proportional to the Laplace-Beltrami operator Δ defined on M:

$$H = \frac{1}{2}\{\mathcal{Q}, \mathcal{Q}^\dagger\} = \frac{1}{2}\left\{d_h d_h^\dagger + d_h^\dagger d_h\right\} = \frac{1}{2}\Delta. \tag{3.3.26}$$

Hence, our supersymmetric ground states must be the harmonic forms

$$\mathcal{H}_{(0)} = \mathcal{H}(M, g) = \bigoplus_{p=0}^{n} \mathcal{H}^p(M, g) = \bigoplus_{p=0}^{n} \mathcal{H}^p_{\text{de Rham}}(M), \tag{3.3.27}$$

where $\mathcal{H}(M, g)$ is the space of harmonic forms on the Riemannian mannifold which can then further be decomposed into the Euclidean sum of harmonic p-forms $\mathcal{H}^p(M, g)$.

[8] This subtle point was pointed out in [25] but was not particularly clear in [2]. In fact, it is possible that is has been confused with the conjugate momenta defined from the classical theory in Eq. (3.3.9).

Otherwise, the fermionic number F and canonical commutation relations are not modified and hence the representation of the Hilbert space remains the same.

We could also similarly define an isomorphic mapping between the bosonic and fermionic subspaces given by $\mathcal{Q}_1 = \mathcal{Q} + \mathcal{Q}^\dagger$ following the discussion in Section 3.2.2. Having found that the harmonic forms are equal to the de Rham cohomology groups which are diffeomorphism invariant, we obtain the result that Q can be expressed in terms of $\mathcal{Q}$

$$\mathcal{Q} = e^{-h} Q e^{h} \tag{3.3.28}$$

and that the $\mathcal{Q}$-complex is isomorphic to the Q-complex.

Further proof: recall in Section 3.2.2that the supersymmetric ground states is shown to correspond to the cohomology of the Q-operator graded by the fermionic number p. Hence, drawing parallels between the earlier discussion and our results in Section 3.2.2, we can then identify the graded Q-cohomology as the de Rham cohomology on M

$$\mathcal{H}^p_{(0)} \cong H^p(\mathcal{Q}) \cong H^p(Q) = H^p_{\text{de Rham}}(M), \tag{3.3.29}$$

while

$$H^p_{\text{de Rham}}(M) \cong \mathcal{H}^p(M, g), \tag{3.3.30}$$

from the refinement of the fermion number p.

Intuitively then, the supersymmetric index is the Euler characteristc of the Q-complex

$$\operatorname{tr}(-1)^F = \sum_{p=0}^{n} (-1)^p \dim H^p(Q) = \sum_{p=0}^{n} (-1)^p \dim H^p_{\text{de Rham}}(Q) = \chi(M). \tag{3.3.31}$$

3.4 Instantons

Given that the action in Eq. (3.3.2) is invariant under supersymmetry, it is possible to employ the localisation principle to determine the Witten index for our non-linear sigma model. Our modus operandi here would be to consider a perturbation theory via a rescaling of the superpotential h: $h \xrightarrow{\lambda >> 1} \lambda h$, such that the partition function localises about the low-lying ground states. Following that, from Morse theory, we expand h about the critical points, keeping our expansion up to first order corrections as in a semi-classical approximation. The higher ordered terms are neglected in the perturbation theory. Let us then denote the set of critical points to be $\{x_1, \dots x_N\}$, and hence,

$$h \cong h(x_i) + \frac{1}{2} \sum_{i=1}^{n} c_i (x^i)^2. \tag{3.4.1}$$

We then carry out the calculation in the neighbourhood of the critical points in assuming that it is locally flat (Euclidean) and that the critical points are isolated. This would simplify our calculations tremendously.

Naturally, we consider the following leading order Hamiltonian at each critical point

$$H^{(0)}(x_i) = \sum_{i=1}^{n} \left(\frac{1}{2} p_i^2 + \frac{1}{2} \lambda^2 c_i (x^i)^2 + \frac{1}{2} \lambda c_i \left[\overline{\psi}^i, \psi^i \right] \right) \tag{3.4.2}$$

in which the curvature dependence vanishes due to the localisation behaviour.

From our results in Section 3.2.3, we would then expect that the perturbative supersymmetric ground states (to leading order correction) can then be expressed as

$$\Psi_i^{(0)} = e^{-\lambda \sum_{i=1}^{n} |c_i| (x^i)^2} \prod_{j: c_j < 0} \overline{\psi}^j \left| 0 \right\rangle , \tag{3.4.3}$$

in which we may identify the number of js in the expression to be the Morse index of h at the critical point x_i. Recall from our discussion in Section 1.3.4 that the Morse index μ_i is the number of negative eigenvalues of the Hessian of the superpotential at x_i.

However, it can be shown that the set of supersymmetric ground states arrived at from this procedure will not remain the same to all orders in perturbation theory. Let us denote Ψ_i to be the corresponding expression of $\Psi_i^{(0)}$ including higher ordered terms. Since the perturbation theory preserves the fermion number symmetry, we will expect that Ψ_j is still a p_i-form just as $\Psi_i^{(0)}$. Denoting V_Ψ to be the set of approximate supersymmetric ground states Ψ_i, the number of exact supersymmetric ground states $V_{\text{exact}} \subseteq V_\Psi$, i.e. not all of Ψ_i are exact supersymmetric ground states annihilated by the Hamiltonian in the full theory. Non-perturbative corrections may correct for some of this degeneracy.

3.4.1 Proof of the Weak Morse Inequality

In [1], Witten expressed this in terms of an eigenvalue problem; the nth smallest eigenvalues of the H_λ acting on p-forms are asymptotically expanded in orders of $1/\lambda$:

$$\varepsilon_p^{(n)}(\lambda) = \lambda \left(A_p^{(n)} + \frac{B_p^{(n)}}{\lambda} + \frac{C_p^{(n)}}{\lambda^2} + \dots \right), \tag{3.4.4}$$

where

$$H_\lambda = \{ d\, d^\dagger + d^\dagger\, d \} + \lambda^2 (dh)^2 + \frac{\lambda}{2} \sum_{i,j} \frac{\partial^2 h}{\partial \phi^i \partial \phi^j} \left[\overline{\psi}^i, \psi^j \right]. \tag{3.4.5}$$

The expression is exact and obtained by considering the localisation principle such that the curvature terms are dropped. $A_p^{(n)}$, $B_p^{(n)}$ and $C_p^{(n)}$ in Eq. (3.4.4) are simply expansion coefficients. Witten argues that b_r is equal to the number of times $\varepsilon_p^{(n)}(\lambda)$ is equal to zero and that this number is bounded above by the number of vanishing $A_p^{(n)}$. We can check that the expansion makes sense by taking λ to the limit of ∞ to see that the higher ordered terms naturally fall off as expected.

Nonetheless, what is certain is that the number of critical points places an upper bound to the number of exact supersymmetric ground states Ψ_j. This translates to the weak Morse inequality described in Eq. (1.3.36).

$$M_r \geq b_r. \tag{Eq. (1.3.36)}$$

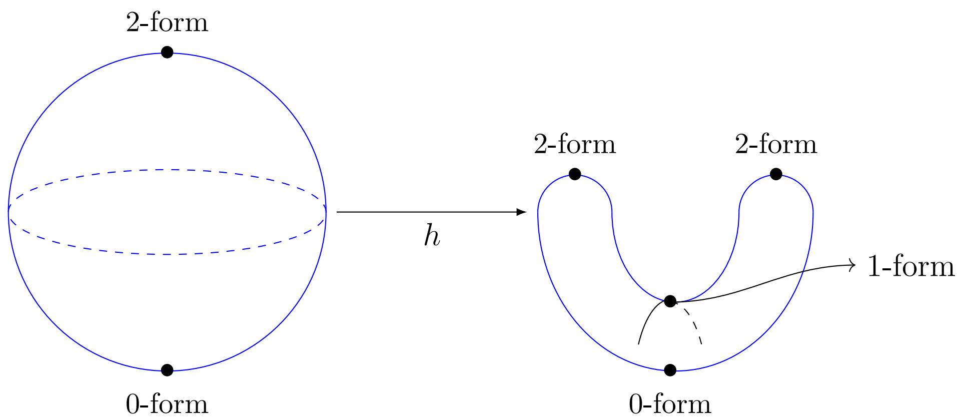


Figure 3.6: Shown here, the deformation of the sphere by the superpotential introduces additional critical points.

3.4.2 Instanton Solutions

This extra counting of supersymmetric ground states can be illustrated by the deformation of a 2-sphere as shown in Fig. 3.6. We introduce the superpotential that deforms the manifold resulting in a somewhat richer structure: we obtain an additional 2-form and 1-form from the deformation. Yet, in the full theory, not all of these states are exact ground states. We may realise that given that both states are bosonic (even numbered) on the undeformed 2-sphere that it would be impossible for a deformation to raise them into excited states. While in the deformed scenario, two linear combinations are exact supersymmetric ground states.

This simple exercise demonstrates our point that it is not necessary the case that all the Ψ_i are exact supersymmetric ground states. Recalling that an exact supersymmetric ground state must be annhilated by $\mathcal{Q}$, for $\Psi_i \in V_\Psi$, it is not necessarily true that

$$\mathcal{Q}\Psi_i = 0 \quad \forall i. \tag{3.4.6}$$

Up to the leading order correction, our expression is modified in the full theory to take the form

$$\mathcal{Q}\Psi_i = \sum_{j=1}^{N} \Psi_j \langle \Psi_j, \mathcal{Q}\Psi_i \rangle \,, \tag{3.4.7}$$

additional terms involving non-zero energy states in the perturbation theory are neglected.

We can see from this expression that Ψ_i is exact only if the correlation function $\langle \Psi_j, \mathcal{Q}\Psi_i \rangle$ is zero. Given that $\mathcal{Q} : \Omega_p(M) \to \Omega_{p+1}(M)$, this means that it is zero if the difference in Morse index is equal or greater than two. This may be rewritten in more concrete terms in terms of the inner product between the two differential forms:

$$\langle \Psi_j, \mathcal{Q}\Psi_i \rangle = \int_M \overline{\Psi_j} \wedge \star(d + dh\wedge)\Psi_i. \tag{3.4.8}$$

If Ψ_j is a μ_j-form and $\mathcal{Q}\Psi_i = \star(d + dh\wedge)\Psi_i$ is a $(\mu_i + 1)$-form, the equation is non-zero only if $\mu_j = \mu_i + 1$. We must hence look for such configurations that allows for what seems to be a tunneling effect. This tunneling mechanism is identified as instantons in the path integral and will lift some of the degeneracy that we spoke of previously. The existence of non-zero matrix elements in Eq. (3.4.8) suggests that the states Ψ_i and Ψ_j are not linearly independent.

Let us put this thought aside for the time being and express the right hand side of the equation in terms of the path integral.

$$\begin{aligned}\langle \Psi_j, \mathcal{Q}\Psi_i\rangle &= \frac{1}{h(x_i)-h(x_j)}\langle \Psi_j, -h(x_j)\mathcal{Q} + \mathcal{Q}h(x_i)\Psi_i\rangle = \frac{1}{h(x_i)-h(x_j)}\langle \Psi_j, [\mathcal{Q},h]\,\Psi_i\rangle\,,\\ &= \frac{1}{h(x_i)-h(x_j)}\lim_{T\to\infty}\langle \Psi_j, e^{-TH}\,[\mathcal{Q},h]\,e^{-TH}\Psi_i\rangle\,.\end{aligned} \tag{3.4.9}$$

The first equality is a leading order in $1/\lambda$ expansion given in [1, 2]; the action of the operator e^{-TH} is inserted to project the states Ψ_i and Ψ_j to the perturbative ground state. We can see this by noting that the Hamiltonian is the generator of time translation, if we were to project a state to $T\to\infty$, there would be no further time translation and the Hamiltonian at $T\to\infty$ is zero. Also note that in the limit of $\lambda\to\infty$, the perturbative ground states are approximately delta functions at x_i. The Morse function h is hence projected to $h(x_i)$ when acting on the ground states. In [3], Witten explains that the exact computation of the left hand side in Eq. (3.4.9) is non-trivial and instead the computation of the reduced maxtrix element of the commutator $[\mathcal{Q},h]$ is suggested as a method to evaluate it classically without dependence in λ.

Hence,

$$\begin{aligned}\langle \Psi_j, \mathcal{Q}\Psi_i\rangle &= \frac{1}{h(x_i)-h(x_j)}\int\limits_{\phi(-\infty)=x_i,\phi(\infty)=x_j}\mathcal{D}\phi\mathcal{D}\overline{\psi}\mathcal{D}\psi\,[\mathcal{Q},f]\,e^{-S_E(\phi,\psi,\overline{\psi})},\\ &= \frac{1}{h(x_i)-h(x_j)}\int\limits_{\phi(-\infty)=x_i,\phi(\infty)=x_j}\mathcal{D}\phi\mathcal{D}\overline{\psi}\mathcal{D}\psi\left(\overline{\psi}^i\frac{\partial h}{\partial x^i}\right)e^{-S_E(\phi,\psi,\overline{\psi})},\end{aligned} \tag{3.4.10}$$

$$\therefore \lim_{T\to\infty}\langle \Psi_j, e^{-TH}\,[\mathcal{Q},h]\,e^{-TH}\Psi_i\rangle = \int\limits_{\phi(-\infty)=x_i,\phi(\infty)=x_j}\mathcal{D}\phi\mathcal{D}\overline{\psi}\mathcal{D}\psi\left(\overline{\psi}^i\frac{\partial h}{\partial x^i}\right)e^{-S_E(\phi,\psi,\overline{\psi})}, \tag{3.4.11}$$

where in the second line we made use of the fact that $[\mathcal{Q},h] = df\wedge = \frac{\partial h}{\partial x^i}\overline{\psi}^i$. Here, we impose the boundary condition that $\phi(-\infty)=x_i, \phi(\infty)=x_j$. The integration is hence carried out over the space with the imposed boundary condition, or to be specific for one pair of critical points x_i, x_j.

It might help to visualise the calculations here: imagine the space on M littered with the various supersymmetric states. The superparticle described in our theory would take on a classical path that minimises the action: on a Riemannian manifold that would naturally be the geodesic. The path integral being the sum over all weighted (Euclidean) paths, suggests that all other semi-classical or quantum mechanical paths are considered in the calculations too. By extending $T \to \infty$, we allow the computation of the path integral to take place virtually over the whole space on M, over the whole space of states. Yet, at the same time, only the low-lying states contribute to the path integral. In practical terms we need only consider T to be large enough such that it encompasses all the critical points of h on M. The condition that our manifold is compact, and that there is a finite number of critical points is hence motivated, though the later condition would be sufficient.

The Euclidean action is obtained by a Wick rotation from $t \to i\tau$ and integrated over $d\tau$ of the classical Lagrangian given previously in Eq. (3.3.2):

$$S_E(\phi, \psi, \overline{\psi}) = \int_{-\infty}^{\infty} d\tau \, \underbrace{\frac{1}{2} g_{ij} \frac{d\phi^i}{d\tau}\frac{d\phi^j}{d\tau} + \frac{1}{2}\lambda^2 g^{ij}\partial_i h \partial_j h}_{L_b} \\ \underbrace{+ \frac{1}{2} g_{ij}\left(\overline{\psi}^i \frac{d\psi^j}{d\tau} + \frac{d\psi^i}{d\tau}\overline{\psi}^j\right) + \frac{\lambda}{2}\partial_i\partial_j h \left[\overline{\psi}^i, \psi^j\right]}_{L_f}, \tag{3.4.12}$$

where we have grouped up the bosonic and fermionic terms in the Lagrangian.

The instanton solutions or tunneling paths are the classical solutions of the Lagrangian ignoring the fermionic terms, i.e. the bosonic action[9]:

$$S_{\text{bosonic}} = \int_{-\infty}^{\infty} d\tau \, \frac{1}{2} g_{ij}\left(\frac{d\phi^i}{d\tau} \pm g^{ik}\partial_k h\right)\left(\frac{d\phi^j}{d\tau} \pm g^{jl}\partial_l h\right) \mp \lambda \int_{-\infty}^{\infty} d\tau \, \frac{dh}{d\tau}, \tag{3.4.13}$$

where we recall that $\partial_i h = \dfrac{\partial h}{\partial \phi^i}$ and hence our expression simplifies to

$$S_{\text{bosonic}} = \int_{-\infty}^{\infty} d\tau \, \frac{1}{2} g_{ij}\left(\frac{d\phi^i}{d\tau} \pm g^{ik}\partial_k h\right)\left(\frac{d\phi^j}{d\tau} \pm g^{jl}\partial_l h\right) \mp \lambda \left(h(x_i) - h(x_j)\right). \tag{3.4.14}$$

From the Principle of Stationary Action, our equations of motion are, as expected, those of the instantons[1–3, 24, 25]:

$$\begin{aligned} \frac{d\phi^i}{d\tau} + g^{ij}\partial_j h = 0, \quad & \text{if}\,(h(x_i) - h(x_j)) < 0, \\ \frac{d\phi^i}{d\tau} - g^{ij}\partial_j h = 0, \quad & \text{if}\,(h(x_i) - h(x_j)) > 0. \end{aligned} \tag{3.4.15}$$

The choice of sign is dependent on the sign of $(h(x_i) - h(x_j))$ as enforced by a positive (Euclidean) action bounded from below. The appropriate choice of sign is determined by our supersymmetric transformation as given in Eq. (3.3.4) in which the partition function localises at regions where the δ_ϵ variation in the fermionic fields is zero, i.e.

$$\frac{d\phi^i}{d\tau} - \lambda g^{ij}\partial_j h = 0. \tag{3.4.16}$$

[9]Instantons are tunneling mechanisms that exist in standard quantum mechanics and in quantum field theories with Abelian gauge, they are not unique to SUSY QM. The difference here is that they gain an additional factor in the path integral due to the additional fermionic degrees of freedom.

We hence obtain a class of solutions for ϕ_i. This result follows from that the integrand $\frac{\partial h}{x^i}\overline{\psi}^i$ is invariant under the ϵ-supersymmetry transformation given in Eq. (3.3.4), it preserves half of the supersymmetry and

$$\delta_\epsilon \left(\frac{\partial h}{\partial x^i} \overline{\psi}^i \right) = 0. \tag{3.4.17}$$

Hence, the partition function would only pick up contributions from the critical points alone and would be zero everywhere else (localisation principle). More specifically it will only pick up contributions from the δ_ϵ-fixed points generated by $\mathcal{Q}$. Hence, we are looking solutions with ascending gradient flow for which $h(x_j) - h(x_i) > 0$. Eq. (3.4.16) is also known as the equation of steepest ascent. The bosonic action $\mathrm{S}_{\text{bosonic}}$ then reduces to

$$S_{\text{bosonic}} = \lambda h(x_j) - h(x_i) > 0. \tag{3.4.18}$$

Let us now look at the fermionic action $S_{\text{fermionic}}$; having a single insertion of $\overline{\psi}^i$in the partition function (Eq. (3.4.11)) places a constraint on the solutions for the fermionic portion.

$$S_{\text{fermionic}} = \int d\tau \, \frac{1}{2} g_{ij} \left(\overline{\psi}^i \frac{d\psi^j}{d\tau} + \frac{d\psi^i}{d\tau} \overline{\psi}^j \right) + \frac{\lambda}{2} \partial_i \partial_j h \left[\overline{\psi}^i, \psi^j \right], \tag{3.4.19}$$

$$= \int d\tau \, \overline{\psi}^i \left(\frac{d\psi^j}{d\tau} + \lambda g^{ij} \partial_j \partial_k h \psi^k \right) = \int d\tau \, g_{ij} \overline{\psi}^i \mathcal{D}_+ \psi^j = -\int d\tau \, g_{ij} \mathcal{D}_- \overline{\psi}^i \psi^j, \tag{3.4.20}$$

where we have re-expressed the differentiation to $\mathcal{D}_\pm \psi^i = \dfrac{d\psi^i}{d\tau} \pm \lambda g^{ij} \partial_j \partial_k h \psi^k$. From this expression and insertion of an additional $\overline{\psi}^i$ in the partition function, the constraint for a non-vanishing partition function is:

$$\operatorname{Ind} \mathcal{D}_- := \dim (\ker D_-) - \dim (\ker D_+) = 1, \tag{3.4.21}$$

which means to that we need an additional zero-mode in the solution to Eq. (3.4.20).

This result is consistent with our discussion earlier in Eq. (3.4.8), and hence

$$\operatorname{Ind} \mathcal{D}_- = \Delta\mu, \tag{3.4.22}$$

where $\Delta\mu = \mu_j - \mu_i$ is the relative Morse index.

3.4.2.1 Genericity assumption

In order to carry out the calculation, we would then need to be able to identify the number of zero-modes in ϕ, ψ and $\overline{\psi}$, given that the excited states would cancel out. In order to do so, we will invoke the assumption that [2]:

$$\ker \mathcal{D}_+ = 0. \tag{3.4.23}$$

This assumption may require rigorous proving but is shown to hold and was mentioned in [3, 26, 27]. Witten seems to suggest this to be a typical feature in topological field theories. We will not go into the proof of this but refer the reader to the relevant papers for a more detailed discussion. Witten argues that in general, $\mathrm{Ind}\,\mathcal{D}_- = \mathcal{M}_{\mathcal{D}_-}$, where we denote $\mathcal{M}_{\mathcal{D}_-}$ to be the moduli space to our instanton solution along the ascending gradient flow lines[10]. Hence, since $\mathrm{Ind}\,\mathcal{D}_- = \Delta\mu = 1$,

$$\mathcal{M}_{\mathcal{D}_-} = 1. \tag{3.4.24}$$

3.4.2.2 Path Integral Evaluation

What this means for evaluating the path integral in Eq. (3.4.11) is that the instanton solution can be parametrised by τ_1:

$$\gamma^i_{\tau_1} = \gamma^i(\tau + \tau_1), \tag{3.4.25}$$

which defines the 'position' of the instanton within the infinite Euclidean time interval. Let us evaluate the path integral $\int \mathcal{D}\phi\mathcal{D}\overline{\psi}\mathcal{D}\psi \left(\overline{\psi}^i \frac{\partial h}{\partial x^i}\right) e^{-S_E(\phi,\psi,\overline{\psi})}$ by making use of the localisation principle. By a change of variables and collecting the terms in evaluating the bosonic and fermionic action

$$S_E(\phi,\psi,\overline{\psi}) = \lambda\left(h(x_j) - h(x_i)\right) + \int d\tau \left[\frac{1}{2}|\mathcal{D}_-\xi|^2 - g_{ij}\mathcal{D}_-\overline{\psi}^i\psi^j\right], \tag{3.4.26}$$

where we expand ϕ about the instantons $\phi = \gamma^i_{\tau_1} + \xi$: ξ is the first order perturbative term about the instanton $\gamma^i_{\tau_1}$. From the generic assumption, we can see that both ϕ and $\overline{\psi}$ have one zero mode while there are no ψ zero mode. This follows from that

$$\mathcal{D}_\pm\delta\phi^i := \nabla_\tau\delta\phi^i \pm \lambda g^{ij}\nabla_j\partial_k\delta\phi^k, \tag{3.4.27}$$

$$\mathcal{D}_\pm\xi^i = \frac{d\xi^i}{d\tau} \pm \lambda g^{ij}\partial_j\partial_k\xi^k. \tag{3.4.28}$$

The nonzero mode path-integral with the action given by Eq. (3.4.26) is a fermionic and bosonic gaussian integral

$$\frac{\det{}'\,(\mathcal{D}_-)}{\sqrt{\det{}'\,\left(\mathcal{D}_-^\dagger\mathcal{D}_-\right)}} = \pm 1. \tag{3.4.29}$$

Intuitively, this follows quite naturally given the pairing of excited states.

The form of the zero mode integral is a familiar result in instanton calculations; it can be expressed and evaluated as follows:

$$\begin{aligned}&\int_{-\infty}^{\infty} d\tau_1 \int d\overline{\psi}_0\overline{\psi}_0 \frac{d\gamma^i_{\tau_1}}{d\tau}\partial_i h|_{\tau=0},\\ &= \int_{-\infty}^{\infty} d\tau_1 \frac{d\gamma^i_{\tau_1}}{d\tau_1}(\tau_1)\partial_i h\left(\gamma(\tau_1)\right) = h(x_j) - h(x_i).\end{aligned} \tag{3.4.30}$$

[10] This is only true if the index is positive. This is said to be the expected or virtual dimension of $\mathcal{M}_{\mathcal{D}_-}$ [27]. If the index of D_- is negative, generically $\mathcal{M}_{\mathcal{D}_-} = 0$

Eq. (3.4.11) hence simplifies to

$$\langle \Psi_j, \mathcal{Q}\Psi_i \rangle = \frac{1}{h(x_i) - h(x_j)} \int \mathcal{D}\phi \mathcal{D}\overline{\psi} \mathcal{D}\psi \left(\overline{\psi}^i \frac{\partial h}{\partial x^i} \right) e^{-S_E(\phi, \psi, \overline{\psi})}, \tag{3.4.31}$$

$$= \sum_\gamma n_\gamma e^{-\lambda(h(x_j) - h(x_i))}. \tag{3.4.32}$$

after summing over the instanton contributions on the manifold M with the Morse index in the pair obeying the constraint $\mu_j = \mu_i + 1$ as previously illustrated.

The value of n_γ is ± 1 depending on the relative orientation between $\overline{\Psi}_j \wedge \star\mathcal{Q}$ and M. The details in determining the orientation are skipped here. More importantly, $n_\gamma = 0$ if the pair of critical points are exact supersymmetric ground states. This happens when we have two instanton solutions connecting Ψ_i and Ψ_j with differing signs[3].

$$\begin{aligned} \mathcal{Q}\Psi_i &= \sum_{j=1}^{N} \Psi_j \langle \Psi_j, \mathcal{Q}\Psi_i \rangle, \\ &= \sum_{j:\mu_j = \mu_i + 1} \Psi_j \sum_\gamma n_\gamma e^{-\lambda(h(x_j) - h(x_i))}. \end{aligned} \tag{3.4.33}$$

3.4.2.3 The Morse-Witten Complex

Using the results here, we are now ready to 'prove' the strong Morse inequalities in Eq. (1.3.33). Since the exponential in Eq. (3.4.11) does not contribute to the summation, it may be treated as a normalisation factor and dropped via an appropriate rescaling in Ψ_j. We can then define the operator

$$\partial \Psi_i := \sum_{j:\mu_j = \mu_i + 1} \Psi_j \sum_\gamma n_\gamma. \tag{3.4.34}$$

It follows that $\partial^2 = 0$ given that $\mathcal{Q}^2 = 0$[11]. We are hence able to define the Morse-Witten complex given by

$$0 \to C^0 \xrightarrow{\partial} C^1 \xrightarrow{\partial} \dots \xrightarrow{\partial} C^n \xrightarrow{\partial} 0, \tag{3.4.35}$$

generated by the graded space of perturbative ground states

$$C^\mu := \bigoplus_{\mu_i = \mu} \mathbb{C}\Psi_i. \tag{3.4.36}$$

The space of supersymmetric ground states is naturally the cohomology group of the Morse-Witten complex. This hence completes our 'proof' for the strong Morse inequalities given by Eq. (1.3.33). The actual details can be found in [1, 3].

[11] A thorough elaboration on why $\mathcal{Q}^2 = 0$ can be found in [2, 3]

Chapter IV

Conclusion

The ability to realise a mathematical theory from a physical system is a rather astonishing step in mathematical physics; where in the past the development of mathematical theories drove the development of physics (e.g. calculus), this relationship between mathematics and physics has been revolutionised in modern day.

In this review, we've managed to acquire an understanding for supersymmetric quantum field theory in zero and one-dimensions. From observing the parallel definitions between quantum mechanics and the one-dimensional quantum field theory, we come to agree that they are equivalent models. In particular, we developed the formalism and applied it to a non-linear sigma model. It was also shown that in quantising the non-linear sigma model, the fermionic fields are realised as differential forms on the Riemannian manifold and give rise to the study of the de Rham cohomology. We also found that the supersymmetric ground states correspond to harmonic forms. This allows us to develop a formalism to study the topology of a manifold from a physical model of nature. The roles may be reverse to study the supersymmetric ground states from the topological properties on the manifold. This duality is the driving force behind the study of topological quantum field theory.

We further showed that Morse theory may be applied to our supersymmetric quantum mechanics model in determining the supersymmetric ground states. The realisation of (weak and strong) Morse inequalities that describes the topology of a manifold was found in our analysis of the supersymmetric ground states in the non-linear sigma model. The number of perturbative supersymmetric ground states was shown to place an upper bound on the number of exact supersymmetric ground states – this result corresponds to the weak Morse inequalities Eq. (1.3.36). We then discussed the indirect proof of the strong Morse inequalities by identifying the Morse-Witten complex on the target space M with the relevant properties.

4.1 Future Work

The review here provided a reasonable exposure and allowed the author to gain the necessary mathematical tools required to understand supersymmetry in higher dimensions. As an extension, the discussion can be extended to study the Landau Ginzburg model in which the bosonic variables are defined on a complex space. It then opens up the possibly to expand look into supersymmetric quantum field theories in (3+1)-dimensions which would give a more realistic model of the world that we live in. An alternative direction in which the author can take up would be to look into thermal quantum field theories which has practical applications in quantum optics, statistical mechanics and quantum open systems.

Certain aspects were also not detailed in the review: verifying the set of supersymmetric variations for the non-linear sigma model discussed in Section 3.3 and calculations involving the instantons in Section 3.4. This would help to verify that the solutions for the conjugate momenta are indeed inaccurate in [2].

Appendix A

One Dimensional QFT: Operator Formalism

The Hilbert space and operator formulation arises when we consider manifolds with boundaries. To each boundary we associate a Hilbert space that corresponds to fixing the field configurations at the boundary. For the one-dimensional case, the boundary would be a point. This can be generalised to say that a n-dimensional manifold M (or n-manifold) has a boundary, ∂M, which is a $(n-1)$-manifold.

Fixing the value of the field at the boundary corresponds to choosing a delta function. The Hilbert space $\mathcal{H}$ in this case is the space of complex-valued square-normalisable functions of the variable X, i.e., $\mathcal{H} = L^2(\mathbb{R};\mathbb{C})$, with the inner product defined as

$$\langle f, g\rangle = \int \overline{f(X)} g(X) dX. \tag{A.0.1}$$

For good reasons, let us recall that $U(t_2, t_1) = e^{-i(t_2-t_1)H}$ is the time evolution unitary operator with H defined to be the Hamiltonian. The unitary operator hence can be seen also as the generator of time evolution. The time evolution operator maps a state $f(X_1)$ at time t_1 to a state $f(X_2)$ at time t_2 between Hilbert spaces,

$$U(t_1, t_2) : \mathcal{H} \to \mathcal{H}. \tag{A.0.2}$$

Operating on the function, $f(X_1)$

$$f(X_1) \to (U_{(t_1,t_2)} f)(X_2) = \int U(t_1, X_1; t_2, X_2) f(X_1) dX_1, \tag{A.0.3}$$

where $f(X_2)$ denotes a different state.

We have shown previously in Eq. (1.4.10) that the Hamiltonian is an invariant quantity, a Noether's charge under time translation symmetry. Hence, it follows that

$$U(t_1, X_1; t_2, X_2) = U(t_1 - t_2, X_1; 0, X_2) =: U_{(t_1-t_2)}(X_1, X_2). \tag{A.0.4}$$

By definition of the property of the unitary operator, we would hence also lead us to find

$$U_{(t_1-t_3)}(X_1, X_3) = \int U_{(t_2-t_3)}(X_2, X_3) U_{(t_1-t_2)}(X_1, X_2) dX_2. \tag{A.0.5}$$

Thes results can be interpreted as saying that the time evolution from t_1 to t_2 and then from t_2 to t_3 is the same as for t_1 to t_3. As mentioned previously in the earlier sections, the Hamiltonian H is a hermitian operator.

One may be quick to draw parallels between the path integral approach arrived in Section 3.1 and notice that the unitary operator defined in this subsection is almost identical. Hence, in Eq. (A.0.5), we may replace U for Z to arrive at

$$Z_{(t_1-t_3)}(X_1, X_3) = \int Z_{(t_2-t_3)}(X_2, X_3) Z_{(t_1-t_2)}(X_1, X_2) dX_2. \tag{A.0.6}$$

We can also see that the unitary operator $Z_{(t_i-t_j)}(X_i, X_j)$ for $i < j$ is dependent on the interval $t_i - t_j$ defined on the manifold M.

Consider the path integral evaluated on the manifold $M = \mathcal{S}^1_\beta$ with circumference β. The Euclidean path-integral on the interval β with the values of X being identical at the initial and final points and integrated over gives

$$Z_E(\beta) = \int dX_1 Z_{E,\beta}(X_1, X_1) = \operatorname{tr}\left\{e^{-\beta H}\right\}. \tag{A.0.7}$$

where we have applied the Wick rotation to arrive at an non-unitary operator. We have hence identified the non-unitary operator as the weight mentioned previously in Eq. (3.1.3). If we were to make the association between β, the inverse temperature, and $-i\tau$ we may notice the similarity between the result obtained here and the familiar thermodynamic partition function for a canonical ensemble. We recall that β here defines the parameter space on the manifold M.

A.1 E.g. Simple Harmonic Oscillator

In order to get a feel of the partition function, we work with the simple harmonic oscillator and work out the results for in the path integral approach. In doing so, we will then compare and show that the results are equivalent to that of in the canonical quantisaton approach.

For the given Euclidean action,

$$Z(\beta) = \int_{X(\tau+\beta)=X(\tau)} DX(\tau) \exp\left(-\int d\tau \left(\frac{1}{2}\dot{X}^2 + \frac{1}{2}X^2\right)\right). \tag{A.1.1}$$

The Euclidean action can be solved by algebraic methods, introducing

$$\frac{1}{2}\int d\tau \left(\frac{1}{2}\dot{X}^2 + \frac{1}{2}X^2\right) = \frac{1}{2}\int d\tau \, X\Theta X \tag{A.1.2}$$

where $\Theta = -\dfrac{d^2}{d\tau^2} + 1$. Let $f_n(t)$ be the orthonormal eigenfunctions of the operator Θ,

$$\Theta f_n(t) = \lambda f_n(t), \quad \int \overline{f}_n(t) f_m(t) = \delta_{n,m}. \tag{A.1.3}$$

We may then expand $X(t)$ in terms of the eigenfunctions to solve the differential equation, $X(t) = \sum_n cnf_n(t)$. This would result in

$$e^{-S} = e^{-\frac{1}{2}\sum_n \lambda_n c_n^2} \tag{A.1.4}$$

$$DX(t) = \prod_n \frac{dc_n}{\sqrt{2\pi}} \tag{A.1.5}$$

where the $^1/_{\sqrt{2\pi}}$ factor arises from normalisation.

The path-integral then becomes

$$Z(\beta) = \prod_n \lambda_n^{-1/2} = \frac{1}{\sqrt{\det(\Theta)}}. \tag{A.1.6}$$

Solving the ODE in Eq. (A.1.2),

$$\lambda_n = 1 + \left(\frac{2\pi n}{\beta}\right)^2, \quad \forall n \in \mathbb{Z}^+. \tag{A.1.7}$$

Note that there are two modes for $n \geq 1$, namely $\sin(2\pi nX/\beta)$ and $\cos(2\pi nX/\beta)$ and only one mode (constant mode) for n = 0. Hence we will need to square each of λ_n for $n \geq 1$ in our expression for Eq. (A.1.6).

$$Z(\beta) = \prod_{n=1}^{\infty} \left(1 + \left(\frac{2\pi n}{\beta}\right)^2\right)^{-1}. \tag{A.1.8}$$

This expression appears to be divergent. Hence, the question now is: how do we make sense of the diverging expression for the partition function.

A.2 Zeta-function Regularisation

Regularisation is a mathematical procedure to make sense of a divergent infinite product. In this process, we replace the infinite result with a finite result in a way so that it keeps the same properties. These finite results can then be used to do calculations and make predictions, as long as the final predictions are independent of the way in which we do regularisation. There's no guarantee that the regularisation would lead us to the correct result and hence only by referencing to experimental calculations can we determine if the procedure is correct. Alternatively, for simple cases such as the simple harmonic oscillator, it is easily verifiable if we were to compare regularisation against the result that one can obtain via the operator formalism.

A way to interpret the regularisation procedure is to understand that the higher energy modes may not be accessible and hence contribute to the actual path integral. It is then necessary to invoke a reasonable cut off to arrive at a realistic answer (experimentally), in preference of a finite value over an unrealistic infinite one. This idea provides further motivation in the formulation of the renormalisation scheme.

We will re-express the partition function to apply the zeta-function regularisation scheme.

$$Z(\beta) = \prod_{n=1}^{\infty} \left(\frac{2\pi n}{\beta}\right)^{-2} \prod_{n=1}^{\infty} \left(1 + \left(\frac{2\pi n}{\beta}\right)^{-2}\right)^{-1}. \tag{A.2.1}$$

The second factor in the expression above is a convergent infinite product and can be shown to be

$$\frac{\beta}{(2\sinh(\beta/2))},$$

while the first factor is a divergent term. This second product can be simplified by identifying it to be $\beta/2 \equiv x$ from Euler's representation of the hyperbolic functions in terms of an infinite product,

$$\sinh(x) = x \prod_{n=1}^{\infty} \left(1 + \left(\frac{x}{n\pi}\right)^{2}\right). \tag{A.2.2}$$

If $0 \leq a_n < 1$, the infinite products $\prod_{n=1}^{\infty}(1+a_n)$ and $\prod_{n=1}^{\infty}(1-a_n)$ converge if $\sum_{n=1}^{\infty} a_n$ converges and diverges if $\sum_{n=1}^{\infty} a_n$ diverges. Equivalently, if $\sum_{n=1}^{\infty} \log a_n$ converges, then the corresponding infinite sum for $\prod_{n=1}^{\infty} a_n$ would also converge.

We will not prove the two theorems but merely use them directly for our cases. Convergence of an infinite summation series would then require some further proofs (either by the ratio, comparison, or integral test).

We focus our attention on the first factor which we suggest is divergent. The factor

$$\prod_{n=1}^{\infty} \left(\frac{2\pi n}{\beta}\right)^{-2}$$

requires a regularisation in which we replace the infinite product by an “equivalent" infinite sum. We shall employ the zeta-function regularisation.

Let's first take a look at the (Riemann) zeta-function:

$$\zeta(s) = \sum_{n\geq 1} n^{-s} \tag{A.2.3}$$

The zeta-function is analytic over the whole s-plane except tat it blows up at the simple pole, $s = 1$.

It is hence possible for us to analytically replace the infinite product to that of a infinite sum that converges. To do so, we consider the **spectral zeta-function**. For

$$\det(\mathcal{O}) = \prod_{n=1} \lambda_n,$$

where λ_n are the eigenvalues of $\mathcal{O}$, we have

$$\log\det(\mathcal{O}) = \mathrm{tr}\{\log \mathcal{O}\} = \sum_{n} \log \lambda_n \tag{A.2.4}$$

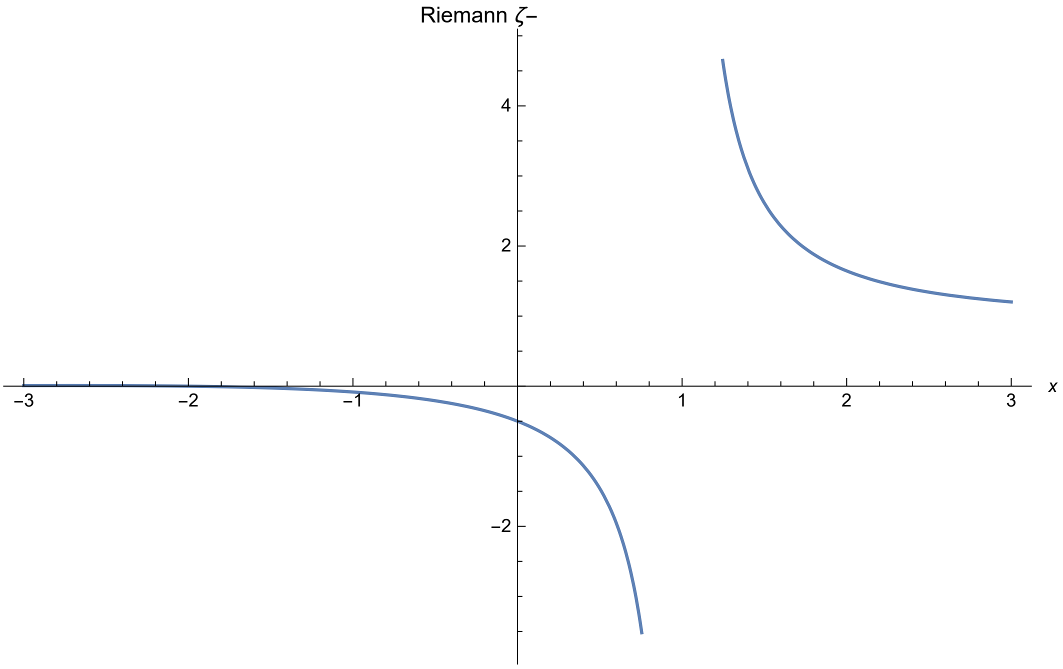


Figure A.1: The Riemann Zeta function is analytic over the whole s-plane except at the simple pole at $s = 1$ in which it blows up.

we then have the spectral zeta-function defined by

$$\begin{aligned} \zeta_{\mathcal{O}}(s) &\equiv \sum_n \frac{1}{\lambda_n{}^s}, \\ \zeta'_{\mathcal{O}}(s=0) &= -\sum_n \log \lambda_n = -\log\left[\prod_{n\geq 1} \lambda_n\right]. \end{aligned} \tag{A.2.5}$$

Hence,

$$\det(\mathcal{O}) = \prod_{n\geq 1} \lambda_n = \exp\left[-\frac{d\zeta_{\mathcal{O}}}{ds}(s=0)\right] \tag{A.2.6}$$

For

$$\prod_{n=1}^{\infty} \left(\frac{2\pi n}{\beta}\right)^{-2},$$

we consider the following summation

$$\zeta_{\mathcal{O}}(s) = \sum_{n\geq 1} \left(\frac{2\pi n}{\beta}\right)^{-2s}, \tag{A.2.7}$$

where

$$\left(\frac{2\pi n}{\beta}\right)^2 = \lambda_n.$$

And hence,

$$\zeta'_{\mathcal{O}}(s=0) = \frac{d}{ds}\left[\sum_{n\geq 1}\left(\frac{2\pi n}{\beta}\right)^{-2s}\right]_{s=0}, \tag{A.2.8}$$

$$= -2\sum_{n\geq 1}\left(\log\left(\frac{2\pi n}{\beta}\right)\right), \tag{A.2.9}$$

$$= \log\left[\prod_{n\geq 1}\left(\frac{2\pi n}{\beta}\right)^{-2}\right]. \tag{A.2.10}$$

Hence, we have

$$\exp\left[\zeta'_{\mathcal{O}}(s=0)\right] = \prod_{n\geq 1}\left(\frac{2\pi n}{\beta}\right)^{-2}. \tag{A.2.11}$$

We would then need to evaluate this summation by differentiating explicitly Eq. (A.2.7).

$$\begin{aligned}\zeta_{\mathcal{O}}(s) &= \sum_{n\geq 1}\left(\frac{2\pi n}{\beta}\right)^{-2s}, \\ &= \left(\frac{\beta}{2\pi}\right)^{2s}\zeta\left(2s\right), \\ \zeta'_{\mathcal{O}}(s=0) &= 2\log\left(\frac{\beta}{2\pi}\right)\zeta(0) + 2\zeta'(0),\end{aligned} \tag{A.2.12}$$

Here then, by noting that

$$\zeta(0) = -\frac{1}{2}, \quad \zeta'(0) = -\frac{1}{2}\log\left(2\pi\right),$$

we find that

$$\zeta'_{\mathcal{O}}(s=0) = -\log\left(\beta\right), \tag{A.2.13}$$

and hence, from Eq. (A.2.8)

$$\prod_{n\geq 1}\left(\frac{2\pi n}{\beta}\right)^{-2} = e^{-\log\beta} = \frac{1}{\beta}. \tag{A.2.14}$$

This exact result may then be substituted into our expression for the partition function in Eq. (A.2.1) to get

$$\begin{aligned}Z\left(\beta\right) &= \frac{1}{\beta}\cdot\frac{\beta}{2\sinh\left(\beta/2\right)} \\ &= \frac{1}{2\sinh\left(\beta/2\right)}.\end{aligned} \tag{A.2.15}$$

This is an identical result that we would also arrive at via the operator formalism, imposing canonical commutation relations on the ladder operators a and $a^\dagger$:

$$Z(\beta) = \mathrm{tr}\left\{e^{-\beta H}\right\} = \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})} = \frac{1}{2\sinh\left(\beta/2\right)}. \tag{A.2.16}$$

And so, while the path integral may appear elegant as it avoids introducing operators and the need to impose canonical commutation relations, the implementation of regularisation seems to be arbitrary. While β is associated as the inverse temperature in the operator formalism, β in the path integral formalism is treated as a (time-related) parameter over which the manifold $M = \mathcal{S}^1$ is defined over. In addition, we have had to introduce the Wick rotation in our evaluation which does not appear in the operator formalism. This of course raises the question of how temperature and time are intepreted in quantum mechanics.

Appendix B

Supersymmetric Harmonic Oscillator

Let us study a case involving a simple harmonic potential. We list and briefly describe its properties here which will be helpful later on. The semi-classical analysis of a supersymmetric harmonic oscillator can be shown to correspond to exact solutions in the supersymmetric framework.

Let us consider a quadratic polynomial for our superpotential such that

$$h(x) = \frac{1}{2}\omega x^2. \tag{B.0.1}$$

The Lagrangian in Eq. (3.2.1) then takes the form

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 - \frac{i}{2}\left(\overline{\psi}\dot{\psi} - \dot{\overline{\psi}}\psi\right) - \omega\overline{\psi}\psi \tag{B.0.2}$$

and the corresponding Hamiltonian to be

$$H = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 + \frac{\omega}{2}\left[\overline{\psi}, \psi\right] \tag{B.0.3}$$

Examining the superpotential $h(x)$ from our previous analysis gives rise to the following results:

$$\Psi = \begin{cases} e^{-\frac{1}{2}\omega x^2}\left|0\right\rangle, & \text{for } \omega > 0, \\ e^{-\frac{1}{2}|\omega|x^2}\overline{\psi}\left|0\right\rangle, & \text{for } \omega < 0. \end{cases} \tag{B.0.4}$$

Recognising that the Hamiltonian describes the sum of a bosonic simple harmonic oscillator and its fermionic counterpart, the spectrum of the total Hamiltonian is given by

$$\begin{array}{ll} \left\{ \begin{array}{l} 0, |\omega|, 2|\omega|, \ldots \\ |\omega|, 2|\omega|, 3|\omega|, \ldots \end{array} \right. & \omega > 0, \\ \left\{ \begin{array}{l} |\omega|, 2|\omega|, 3|\omega|, \ldots \\ 0, |\omega|, 2|\omega|, \ldots \end{array} \right. & \omega < 0. \end{array} \tag{B.0.5}$$

Hence, we verify that there indeed exists a supersymmetric ground state with zero energy, and also that the energies are paired for the excited states. Note that because the superpotential is quadratic in x, we only have one supersymmetric ground state. Even without doing the explicit calculations for the Witten index of the system, it is to be expected that $\mathrm{tr}\,(-1)^F = \pm 1$.

Given that the Hilbert space of the system can be reduced into separate subspaces for bosonic and fermionic parts (due to the absence of interaction terms) we may determine the partition function and Witten index by doing the calculations individually.

$$\begin{aligned} Z(\beta) &:= \mathrm{tr}_{\mathcal{H}}\, e^{-\beta H} = \mathrm{tr}_{L^2}\, e^{-\beta H_{\mathrm{osc}}} \cdot \mathrm{tr}_{\mathbb{C}^2}\, e^{-\beta H_f}, \\ \mathrm{tr}\,(-1)^F &:= \mathrm{tr}_{\mathcal{H}} \left[(-1)^F e^{-\beta H} \right] = \mathrm{tr}_{L^2}\, e^{-\beta H_{\mathrm{osc}}} \cdot \mathrm{tr}_{\mathbb{C}^2} \left[(-1)^F e^{-\beta H_f} \right]. \end{aligned} \tag{B.0.6}$$

$\mathbb{C}^2$ refers to the fermionic space of states while $L^2 := L^2(\mathbb{R}, \mathbb{C})$ refers to the space of square integrable functions for the bosonic harmonic oscillator. H_f is the fermionic Hamiltonian (last term) in Eq. (B.0.3). In matrix representation,

$$H_f = \frac{\omega}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{B.0.7}$$

Then,

$$\begin{aligned} Z(\beta) &= \frac{e^{\frac{\beta\omega}{2}} + e^{-\frac{\beta\omega}{2}}}{e^{\frac{\beta|\omega|}{2}} - e^{-\frac{\beta|\omega|}{2}}} = \coth\left(\beta\omega/2\right), \\ \mathrm{tr}\,(-1)^F &= \frac{e^{\frac{\beta\omega}{2}} - e^{-\frac{\beta\omega}{2}}}{e^{\frac{\beta|\omega|}{2}} - e^{-\frac{\beta|\omega|}{2}}} = \frac{\omega}{|\omega|} = \pm 1. \end{aligned} \tag{B.0.8}$$

An important result is found for the Witten index: the Witten index is independent of the circumference β of S^1. This is what we spoke of previously in Eqs. (3.2.42) and (3.2.43) of Section 3.2.2. This has further physical implications: given that there is β independence for the Witten index, this means that the evaluation of the path integral can be reduced to the zero-dimensional form. Only time independent modes would contribute to the integral. This is an important feature in supersymmetric theories in which we reduce a higher dimensional integral to a lower one. This also suggests that the Witten index is equal to the partition function of the zero-dimensional supersymmetric QFT system discussed in Section 2.2.

From this simple example, we can also see the properties of Morse theory in upersymmetric QM. Recall that the ω in Eq. (B.0.7) can be expressed in terms of the Hessian (matrix) of the superpotential $h(x)$ at the critical point $x = 0$, we can see that the number of bosonic supersymmetric ground states (states that are annihilated by ψ) is equal to the number of positive eigenvalues n_+ of the Hessian. A similar analysis can be made by examining the number of fermionic supersymmetric ground states, which we can show to correspond to the number of negative eigenvalues n_- of the Hessian [3]. Except in this case, the Hessian is single-valued, solely dependent on the sign of ω. However, this analysis can be generalised, as we was shown in the main text, the Witten index (recall

Eq. (3.2.39)) is found by considering the difference in the number of positive and negative eigenvalues of the Hessian:

$$\begin{aligned} \mathrm{tr}\,(-1)^F &= n_+ - n_-, \\ &= n_+ - \mu, \end{aligned} \tag{B.0.9}$$

where μ is the Morse index of an arbitrary Morse function.

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