

**Question 1(a)**

Find the slope of the tangent to the curve  $y^2 = x^3 + 2x^2 - 20$  at the point  $(3, 5)$ .

$$y^2 = x^3 + 2x^2 - 20$$

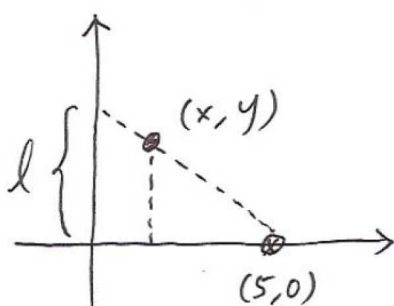
$$2yy' = 3x^2 + 4x$$

$$x = 3, y = 5 \Rightarrow 10y' = 27 + 12 = 39$$

$$\therefore y' = \frac{39}{10}$$

**Question 1(b)**

A lamp is located at the point  $(5, 0)$  in the  $xy$ -plane. An ant is crawling in the first quadrant of the plane and the lamp casts its shadow onto the  $y$ -axis. How fast is the ant's shadow moving so that its  $x$ -coordinate is increasing at a rate of  $\frac{1}{2}$  units/sec and its  $y$ -coordinate is decreasing at a rate of  $\frac{1}{5}$  units/sec?



$$\frac{l}{y} = \frac{5}{5-x} \Rightarrow l = \frac{5y}{5-x}$$

$$\frac{dl}{dt} = \frac{5 \frac{dy}{dt} (5-x) + 5y \frac{dx}{dt}}{(5-x)^2}$$

$$x = 1, y = 2 \Rightarrow \frac{dx}{dt} = \frac{1}{2}, \quad \frac{dy}{dt} = -\frac{1}{5}$$

$$\therefore \frac{dl}{dt} = \frac{5 \left(-\frac{1}{5}\right) (5-1) + 5(2) \left(\frac{1}{2}\right)}{(5-1)^2} = \frac{1}{16}$$

**Question 2(a)**

Find the exact value of the integral

$$\int_0^{\sqrt{101}} 2x^3 e^{x^2} dx.$$

$$u = x^2, \quad du = 2x dx$$

$$\therefore \int_0^{\sqrt{101}} 2x^3 e^{x^2} dx = \int_0^{101} u e^u du = [u e^u]_0^{101} - \int_0^{101} e^u du = 101 e^{101} - [e^u]_0^{101} = 101 e^{101} - 1$$

### Question 2(b)

Find a degree 3 polynomial to approximate the function  $f(x) = \ln(1 + \sin x)$  near  $x = 0$ .

$$f(x) = \ln(1 + \sin x) \Rightarrow f(0) = 0$$

$$f'(x) = \frac{\cos x}{1 + \sin x} \Rightarrow f'(0) = 1$$

$$f''(x) = \frac{-\sin x (1 + \sin x) - \cos^2 x}{(1 + \sin x)^2} = -\frac{1}{1 + \sin x} \Rightarrow f''(0) = -1$$

$$f'''(x) = \frac{\cos x}{(1 + \sin x)^2} \Rightarrow f'''(0) = 1$$

$$\therefore f(x) \approx 0 + x - \frac{1}{2}x^2 + \frac{1}{6}x^3$$

### Question 3(a)

Let  $f(x) = |\sin x|$  for all  $x \in (-\pi, \pi)$  and  $f(x + 2\pi) = f(x)$  for all  $x$ . Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier Series which represents  $f(x)$ . Let  $m$  denote a fixed positive integer. Find the exact value of  $a_{2m}$ .

$f$  is even.

$$\begin{aligned} \therefore a_{2m} &= \frac{2}{\pi} \int_0^{\pi} \sin x \cos 2mx dx \\ &= \frac{1}{\pi} \int_0^{\pi} \sin(2m+1)x - \sin(2m-1)x dx \\ &= \frac{1}{\pi} \left[ -\frac{1}{2m+1} \cos(2m+1)x + \frac{1}{2m-1} \cos(2m-1)x \right]_0^{\pi} \\ &= \frac{1}{\pi} \left\{ \frac{(-1)^{2m+2}}{2m+1} + \frac{1}{2m+1} + \frac{(-1)^{2m-1}}{2m-1} - \frac{1}{2m-1} \right\} \end{aligned}$$

$$= \frac{1}{\pi} \left( \frac{2}{2m+1} - \frac{2}{2m-1} \right)$$

$$= -\frac{4}{(4m^2-1)\pi}$$

### Question 3(b)

Find the shortest distance from the point  $(-1, 1, 2)$  to the plane  
 $2x + 3y - z - 10 = 0$ .

$$d = \frac{|2(-1) + 3(1) - 2 - 10|}{\sqrt{4 + 9 + 1}} = \frac{11}{\sqrt{14}}$$

### Question 4(a)

Let  $L_1$  be a straight line which passes through the point  $(-1, 0, 1)$  and suppose that  $L_1$  is perpendicular to the plane  $2x - y + 7z = 12$ . Let  $L_2$  be the line  $\vec{r}(t) = (2+t)\hat{i} + (-4+2t)\hat{j} + (18-3t)\hat{k}$ . Find the coordinates of the point of intersection of  $L_1$  and  $L_2$ .

$$L_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix} = \begin{pmatrix} -1+2s \\ -s \\ 1+7s \end{pmatrix}, \quad L_2 = \begin{pmatrix} 2+t \\ -4+2t \\ 18-3t \end{pmatrix}$$

$$\begin{pmatrix} -1+2s \\ -s \\ 1+7s \end{pmatrix} = \begin{pmatrix} 2+t \\ -4+2t \\ 18-3t \end{pmatrix} \Rightarrow t = 1$$

$\therefore$  Point of intersection,  $(3, -2, 15)$ .

### Question 4(b)

Let  $f(x, y) = \ln(\tan x + \tan y)$ , with  $0 < x, y < \frac{\pi}{2}$ . Find the value of

$$(\sin 2x) \frac{\partial f}{\partial x} + (\sin 2y) \frac{\partial f}{\partial y}.$$

$$\frac{\partial f}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y}, \quad \frac{\partial f}{\partial y} = \frac{\sec^2 y}{\tan x + \tan y}$$

$$\therefore \sin 2x \frac{\partial f}{\partial x} + \sin 2y \frac{\partial f}{\partial y} = \frac{2 \tan x}{\tan x + \tan y} + \frac{2 \tan y}{\tan x + \tan y} = 2$$

**Question 5(a)**

Let  $n$  be a positive integer. Find the directional derivative of

$$f(x, y) = x^2 - xy + y^n$$

at the point  $(2, 1)$  in the direction of the vector joining the point  $(2, 1)$  to the point  $(6, 4)$ .

$$\vec{u} = \frac{\begin{pmatrix} 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}}{\left| \begin{pmatrix} 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right|} = \frac{1}{5} \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} 2x - y \\ -x + ny^{n-1} \end{pmatrix} \Rightarrow \nabla f(2, 1) = \begin{pmatrix} 3 \\ n - 2 \end{pmatrix}$$

$$\therefore D_{\vec{u}}f(2, 1) = \nabla f(2, 1) \cdot \vec{u} = \frac{12}{5} + \frac{3(n - 2)}{5} = \frac{3n + 6}{5}$$

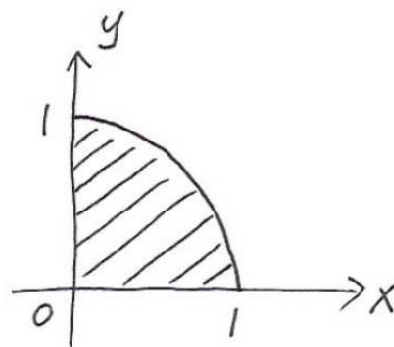
**Question 5(b)**

Evaluate

$$\iint_D x \, dA,$$

where  $D$  is the finite plane region in the first quadrant bounded by the 2 coordinate axes and the curve  $y = 1 - x^2$ .

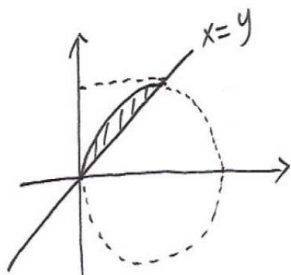
$$\begin{aligned} \therefore \iint_D x \, dx dy &= \int_0^1 \int_0^{1-x^2} x \, dy \, dx \\ &= \int_0^1 [xy]_0^{1-x^2} dx \\ &= \int_0^1 x - x^3 \, dx \\ &= \left[ \frac{1}{2}x^2 - \frac{1}{4}x^4 \right]_0^1 \\ &= \frac{1}{4} \end{aligned}$$



**Question 6(a)**

Find the exact value of the integral

$$\int_0^1 \int_{1-\sqrt{1-y^2}}^y y e^{x^2 - \frac{2}{3}x^3} dx dy.$$



$$x = 1 - \sqrt{1 - y^2} \Leftrightarrow y = \sqrt{2x - x^2}$$

$$\begin{aligned} \therefore \int_0^1 \int_{1-\sqrt{1-y^2}}^y y e^{x^2 - \frac{2}{3}x^3} dx dy &= \int_0^1 \int_x^{\sqrt{2x-x^2}} y e^{x^2 - \frac{2}{3}x^3} dx dy \\ &= \int_0^1 (2x - x^2) e^{x^2 - \frac{2}{3}x^3} dx \\ &= \left[ \frac{1}{2} e^{x^2 - \frac{2}{3}x^3} \right]_0^1 \\ &= \frac{1}{2} \left( e^{\frac{1}{3}} - 1 \right) \end{aligned}$$

**Question 6(b)**Let  $a$  be a positive constant. Evaluate the line integral

$$\int_C x^2 + y^2 + z^2 ds,$$

where  $C$  is the circular helix given by  $x = a \cos t, y = a \sin t, z = t, 0 \leq t \leq a$ .

$$C : \vec{r}(t) = \begin{pmatrix} a \cos t \\ a \sin t \\ t \end{pmatrix} \Rightarrow \vec{r}'(t) = \begin{pmatrix} -a \sin t \\ a \cos t \\ 1 \end{pmatrix}$$

$$||\vec{r}'(t)|| = \sqrt{1 + a^2}$$

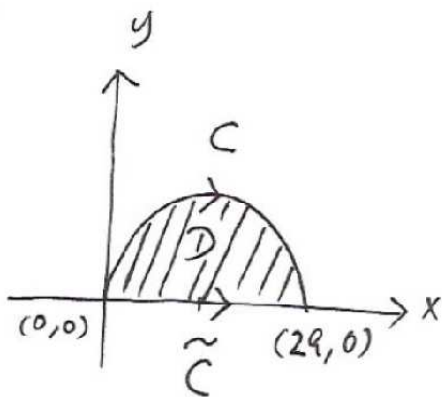
$$\therefore \int_C x^2 + y^2 + z^2 ds = \int_0^a (a^2 + t^2) \sqrt{1 + a^2} dt = \sqrt{1 + a^2} \left[ a^2 t + \frac{1}{3} t^3 \right]_0^a = \frac{4}{3} a^3 \sqrt{1 + a^2}$$

**Question 7(a)**

Let  $a$  be a positive constant. Evaluate the line integral

$$\int_C (2xe^{\sin y} + 3x^2y^2 + ay)dx + (x^2e^{\sin y}\cos y + 2x^3y + 2ax + 1)dy,$$

where  $C$  is the semicircle, centered at  $(a, 0)$  with radius  $a$ , in the first quadrant joining  $(0, 0)$  to  $(2a, 0)$ .



$$\tilde{C} : \vec{r}(t) = (t, 0), \quad 0 \leq t \leq 2a$$

$$\partial D = \tilde{C} - C$$

Apply Green's Theorem to  $D$ ,

$$P = 2xe^{\sin y} + 3x^2y^2 + ay$$

$$Q = x^2e^{\sin y}\cos y + 2x^3y + 2ax + 1$$

$$\oint_{\partial D} P dx + Q dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$$

$$= \iint_D (2xe^{\sin y}\cos y + 6x^2y + 2a - 2xe^{\sin y}\cos y - 6x^2y - a) dA$$

$$= \iint_D a dA$$

$$= a \times \text{area } D$$

$$= \frac{1}{2}\pi a^3$$

$$\int_{\tilde{C}} P dx + Q dy - \int_C P dx + Q dy = \frac{1}{2}\pi a^3$$

$$\therefore \int_C P dx + Q dy = \int_{\tilde{C}} P dx + Q dy - \frac{1}{2}\pi a^3 = \int_0^{2a} 2t dt - \frac{1}{2}\pi a^3 = 4a^2 - \frac{1}{2}\pi a^3$$

**Question 7(b)**

Evaluate the surface integral

$$\iint_S \vec{F} \cdot d\vec{S},$$

where  $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$  and  $S$  is the portion of the paraboloid  $z = 1 - x^2 - y^2$  lying on and above the  $xy$  plane. The orientation of  $S$  is given by the outer normal vector.

$$S : \vec{r}(u, v) = \begin{pmatrix} u \\ v \\ 1 - u^2 - v^2 \end{pmatrix}$$

$$\vec{r}_u = \begin{pmatrix} 1 \\ 0 \\ -2u \end{pmatrix}, \quad \vec{r}_v = \begin{pmatrix} 0 \\ 1 \\ -2v \end{pmatrix}$$

$$\vec{r}_u \times \vec{r}_v = 2u\hat{i} + 2v\hat{j} + \hat{k}$$

At  $(0,0,1)$ ,  $\vec{r}_u \times \vec{r}_v = \vec{k}$  points outwards.

$$\begin{aligned} \therefore \iint_S \vec{F} \cdot d\vec{S} &= \iint_{u^2+v^2 \leq 1} (2u^2 + 2v^2 + 1 - u^2 - v^2) du dv \\ &= \int_0^{2\pi} \int_0^1 r(1 + r^2) dr d\theta \\ &= 2\pi \left[ \frac{1}{2}r^2 + \frac{1}{4}r^4 \right]_0^1 \\ &= \frac{3\pi}{2} \end{aligned}$$

**Question 8(a)**

By using Stokes' Theorem or otherwise, find the exact value of the surface integral

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

where  $S$  is the hemisphere  $x^2 + y^2 + z^2 = 16$  lying on and above the  $xy$  plane, and  $\vec{F} = (x^2 + y - 4e^z)\hat{i} + (3xy \cos^2 z)\hat{j} + (2e^{xy} \sin z + x^2 y z^3)\hat{k}$ . The orientation of  $S$  is given by the outer normal vector.

$$C : \vec{r} = 4 \cos t \hat{i} + 4 \sin t \hat{j}, \quad 0 \leq t \leq 2\pi$$

Note that the orientation of  $C$  is anti-clockwise and this matches with the outer normal orientation of  $S$ .

$\therefore$  By Stoke's Theorem,

$$\begin{aligned}
 \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} &= \int_C \vec{F} \cdot d\vec{r} \\
 &= \int_0^{2\pi} (16 \cos^2 t + 4 \sin t - 4)(-4 \sin t) + 48 \sin t \cos t (4 \cos t) dt \\
 &= \int_0^{2\pi} 128 \cos^2 t \sin t - 16 \sin^2 t + 16 \sin t dt \\
 &= \left[ -\frac{128}{3} \cos^3 t \right]_0^{2\pi} - 8 \int_0^{2\pi} 1 - \cos 2t dt \\
 &= -16\pi
 \end{aligned}$$

### **Question 8(b)**

**Find a solution of the form  $u(x, y) = F(ax + y)$ , where  $a$  is a constant and  $F$  is a differentiable single variable function, to the partial differential equation**

$$u_x - 2u_y = 0,$$

**that satisfies the condition  $u(x, 0) = \cos x$ .**

$$u_x = aF'(ax + y)$$

$$u_y = F'(ax + y)$$

$$u_x - 2u_y = 0 \Rightarrow aF'(ax + y) - 2F'(ax + y) = 0 \Rightarrow a = 2$$

$$u(x, y) = F(2x + y)$$

$$u(x, 0) = \cos x \Rightarrow F(2x) = \cos x \Rightarrow F(x) = \cos \frac{x}{2}$$

$$\therefore u(x, y) = F(2x + y) = \cos \frac{2x + y}{2}$$