PC 3231 - Electricity and Magnetism 2

AY05/06 SEM 1
Suggested Solutions

Q1

(i)

Within the coaxial cable, drawing an Amperian loop of radius $a < s < b$ coaxial with the cable,

$$\begin{aligned}
& \int \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{a} \\
\Rightarrow \quad & 2\pi s B = \mu_0 I \\
\Rightarrow \quad & \vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi}
\end{aligned}$$

Drawing a cylindrical gaussian surface of radius $a < s < b$ with its axis coinciding with that of the coaxial cable,

$$\begin{aligned}
& \int \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} \int \rho d\tau \\
\Rightarrow \quad & (2\pi s l E) = \frac{\lambda l}{\epsilon_0} \\
\Rightarrow \quad & \vec{E} = \frac{\lambda}{2\pi s \epsilon_0} \hat{s}
\end{aligned}$$

Then,

$$\begin{aligned}
\vec{S} &= \frac{1}{\mu_0} \vec{E} \times \vec{B} \\
&= \frac{I\lambda}{4\pi^2 s^2 \epsilon_0} \hat{z}
\end{aligned}$$

(ii)

$$\begin{aligned}
P &= \int \vec{S} \cdot d\vec{a} \\
&= \frac{I\lambda}{4\pi\epsilon_0} \int_a^b \frac{1}{s^2} (2\pi s)\ ds \\
&= \frac{I\lambda}{2\epsilon_0} \ln \frac{b}{a}
\end{aligned}$$

(iii)

$$
\begin{aligned}
\vec{p}_{em} &= \mu_0\epsilon_0 \int \vec{S} d\tau \\
&= \frac{\mu_0\epsilon_0 I\lambda L}{4\pi} \int_a^b \frac{1}{s^2\epsilon_0}(2\pi s)\ ds\hat{z} \\
&= \frac{\mu_0 \lambda I L}{2\pi} \ln\left(\frac{b}{a}\right)\hat{z}
\end{aligned}
$$

(iii)

Drawing a retangular Amperian loop with its normal in the $\hat{\phi}$ direction, one edge of length l at the middle of the coaxial cable and the opposite edge at $a < s < b$,

$$
\begin{aligned}
\int \vec{E}\cdot d\vec{l} &= -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{a} \\
\Rightarrow \quad El &= -\frac{\mu_0}{2\pi}\frac{dI}{dt}\int_a^s \frac{1}{s}(lds) \\
\Rightarrow \quad \vec{E} &= -\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln\frac{s}{a}\hat{z}
\end{aligned}
$$

(iv)

For the $-\lambda$ at $s = b$,

$$
\begin{aligned}
\vec{F} &= q\vec{E} \\
&= (-\lambda L)\left(-\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln\frac{b}{a}\right)\hat{z} \\
&= \frac{\lambda\mu_0 L}{2\pi}\frac{dI}{dt}\ln\frac{b}{a}\hat{z}
\end{aligned}
$$

For the λ at $s = a$, $\vec{E} = 0$ and $\vec{F} = 0$.

(v)

$$
\begin{aligned}
\vec{P} &= \int \vec{F}\ dt \\
&= \left[\frac{\lambda L\mu_0}{2\pi}\ln\frac{b}{a}\int\frac{dI}{dt}\ dt\right]\hat{z} \\
&= \frac{\lambda L\mu_0}{2\pi}\ln\frac{b}{a}(I-0)\hat{z}
\end{aligned}
$$

which is precisely the momentum that was originally stored in the fields.

Q2

(A)

Consider 2 sets of potential:

$$\begin{aligned}\vec{A}' &= \vec{A} + \vec{\alpha} \\ V' &= V + \beta\end{aligned}$$

such that $\vec{A}$ and $\vec{A}'$ give the same $\vec{B}$ and $\vec{E}$:

$$\begin{aligned}\vec{B} &= \nabla \times \vec{A} = \nabla \times \vec{A}' \\ \nabla \times \vec{\alpha} = 0.\end{aligned}$$

Writing α as the gradient of a scalar λ,

$$\begin{aligned}\alpha &= \nabla\lambda \; (\nabla \times \nabla\lambda = 0) \\ \vec{E} &= -\nabla V - \frac{\partial}{\partial t}\vec{A} \\ &= -\nabla V' - \frac{\partial}{\partial t}\vec{A}'\end{aligned}$$

hence

$$\begin{aligned}\nabla\beta + \frac{\partial}{\partial t}\vec{\alpha} &= 0 \\ \nabla(\beta + \frac{\partial}{\partial t}\lambda) = 0\end{aligned}$$

The term in parantheses is independent of position, but it could depend on time :

$$\beta = -\frac{\partial}{\partial t} + k(t)$$

We can absorb $k(t)$ into λ, without affecting the gradient. Hence,

$$\vec{A}' = \vec{A} + \nabla\lambda,$$
$$V' = V - \frac{\partial}{\partial t}\lambda.$$

We can add $\nabla\lambda$ to $\vec{A}$, provided we simulatenously subtract $\vec{\partial}\partial t\lambda$ from V.

(ii)

$$\begin{aligned}\vec{E} &= -\nabla V - \frac{\partial}{\partial t}\vec{A} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \\ \vec{B} &= \nabla \times \vec{A} = 0\end{aligned}$$

This is a set of potentials for a stationary point charge q at the origin, more usually

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0}\frac{q}{r} \\ \vec{A} &= 0. \end{aligned}$$

(iii)

Gauge transforming by λ, we have

$$\begin{aligned} V' &= V - \frac{\partial}{\partial t}\lambda = 0 - \left(-\frac{1}{4\pi\epsilon_0}\frac{q}{r}\right) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \\ \vec{A}' &= \vec{A} + \nabla\lambda = -\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{r} + \frac{-1}{4\pi\epsilon_0}qt\frac{-1}{r^2}\hat{r} = 0 \end{aligned}$$

as in 'usual' potentials of a point charge.

(B)

For convenience, let's say the particle passes through the origin at time $t = 0$, so that

$$\vec{w}(t) = \vec{v}t$$

. We first compute the retarded time :

$$|\vec{r} - \vec{v}t_r| = c(t - t_r)$$

, or squaring:

$$r^2 - 2\vec{r}\cdot\vec{v}t_r + v^2t_r^2 = c^2(t^2 - 2tt_r + t_r^2)$$

Solving for t_r, we find that

$$t_r = \frac{(c^2t - \vec{r}\cdot\vec{v} \pm \sqrt{(c^2t - \vec{r}\cdot\vec{v})^{+}(c^2 - v^2)(r^2 - c^2t^2)}}{c^2 - v^2}$$

To fix the sign, consider the limit $v = 0$:

$$t_r = t \pm \frac{r}{c}.$$

In this case the charge is at rest at the origin, and the retarded time should be $(t - r/c)$; evidently we want the minus sign. Now,

$$\begin{aligned} \mathrm{r} &= c(t - t_r) \\ \vec{\mathrm{r}} &= \frac{\vec{r} - \vec{v}t_r}{c(t - t_r)} \end{aligned}$$

, so

$$\begin{aligned}
\mathrm{r}(1-\hat{r}\cdot\vec{v}/c) &= c(t-t_r)\left[1-\frac{\vec{v}}{c}\cdot\frac{(\vec{r}-\vec{v}t_r)}{c(t-t_r)}\right] = c(t-t_r)-\frac{\vec{v}\cdot\vec{r}}{c}-\frac{v^2}{c}t_r \\
&= \frac{1}{c}[(c^2t-\vec{r}\cdot\vec{v})-(c^2-v^2)t_r] \\
&= \frac{1}{c}\sqrt{(c^2t-\vec{r}\cdot\vec{v})+(c^2-v^2)(r^2-c^2t^2)}
\end{aligned}$$

Therefore,

$$V(\vec{r},t)=\frac{1}{4\pi\epsilon_0}\frac{qc}{\sqrt{(c2t-\vec{r}\cdot\vec{v})^2+(c^2-v^2)(r^2-c^2t^2)}}$$

and

$$\vec{A}(\vec{r},t)=\frac{\mu_0}{4\pi\epsilon_0}\frac{qc\vec{v}}{\sqrt{(c2t-\vec{r}\cdot\vec{v})^2+(c^2-v^2)(r^2-c^2t^2)}}.$$

Q3

(i)

$$\begin{aligned}
\nabla V &= \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} \\
&= -\frac{p_0\omega}{4\pi\epsilon_0 c}\left\{\cos\theta[-\frac{1}{r^2}\sin\omega(t-r/c) - \frac{\omega}{rc}\cos\omega(t-r/c)] - \frac{\sin\theta}{r^2}\sin\omega(t-r/c)\hat{\theta}\right\} \\
&\approx \frac{p_0\omega^2}{4\pi\epsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\cos\omega(t-r/c)\hat{r},
\end{aligned}$$

$$\frac{\partial}{\partial t}\vec{A} = -\frac{\mu_0 p_0 \omega^2}{4\pi r}\cos[\omega(t-r/c)](\cos\theta\hat{r} - \sin\theta\hat{\theta}),$$

so

$$\begin{aligned}
\vec{E} &= -\nabla V - \frac{\partial \vec{A}}{\partial t} \\
&= -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\omega(t-r/c)\hat{\theta}.
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
\vec{B} &= \nabla \times \vec{A} \\
&= \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{\phi} \\
&= \frac{-\mu_0 q_0 \omega}{4\pi r}\left\{\frac{\omega}{c}\sin\theta\cos\omega(t-r/c) + \frac{\sin\theta}{r}\sin\omega(t-r/c)]\right\}\hat{\phi} \\
&\approx -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-\frac{r}{c}]\hat{\phi}.
\end{aligned}$$

(ii)

$$\begin{aligned}
<\vec{S}> &= \frac{1}{\mu_0}(<\vec{E}\times\vec{B}>) \\
&= \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right) < \cos\omega(t-r/c) >\right\}^2\hat{r} \\
&= \left(\frac{\mu_0 p_0^4 \omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\hat{r}.
\end{aligned}$$

(iv)

$$
\begin{aligned}
< P > &= \int < \vec{S} > \cdot d\vec{a} \\
&= \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \int \frac{\sin^2\theta}{r^2} r^2 \sin\theta d\theta d\phi \\
&= \frac{\mu_0 p_0^2 \omega^4}{12\pi c}
\end{aligned}
$$

(v)

$$P = I^2 R = q_0^2 \omega^2 \sin^2 \omega t R$$

Average power,

$$< P >= \frac{1}{2} q_0^2 \omega^2 R$$

Equating this to the power of a dipole,

$$< P >= \frac{\mu_0 q_0^2 \omega^4 d^2}{12\pi c}$$

,

$$R = \frac{\mu_0 d^2}{6\pi c}\omega^2 = \frac{\mu_0 d^2}{6\pi c}\frac{4\pi^2 c}{\lambda^2} = \frac{2}{3}\pi\mu_0 c \left(\frac{d}{\lambda}\right)^2$$

Q4

Inside the sphere, the potential can have no $\frac{1}{r^{l+1}}$ terms, or else it will blow up and die at $r = 0$, so

$$V_in = \sum_0^\infty (A_l r^l) \mathrm{P}_\mathrm{l}(\cos\theta)$$

Outside the sphere, there can be no r^l term, or else it will blow up at $r = \infty$, so

$$V_{out} = \sum_0^\infty \frac{B_l}{r^(l+1)} \mathrm{P}_\mathrm{l}(\cos\theta)$$

.

Let the polarisation of the sphere be in the z direction, so that the bound charge $\sigma_b = \vec{P} \cdot \hat{n} = P\cos\theta$.

The fact that the normal components of E suffer a discontinuity of $\frac{\sigma}{\epsilon}$ at the boundary between the inside and outside of the sphere means that

$$\left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\right)_{r=R} = -\frac{\sigma_b}{\epsilon_0}$$
$$\Rightarrow \quad \sum (l+1)\frac{B_l}{r^{l+1}} \mathrm{P}_\mathrm{l}(\cos\theta) - lA_l R^{l-1} \mathrm{P}_\mathrm{l}(\cos\theta) = -\frac{P\cos\theta}{\epsilon_0}.$$

Since there is only a $\cos\theta = \mathrm{P}_1(\cos\theta)$ term on the right hand side of the equal sign, by the orthogonality of the Legendre polynomials, there can only be P_1 terms on the left hand side of the equal sign as well, so $l = 1$ is the only admissable value. This leads to the simplification that

$$\begin{aligned} V_{out} &= \frac{B}{r^2}\cos\theta \\ V_{in} &= Ar\cos\theta \end{aligned}$$

, and the boundary condition about the discontinuty of the normal component of $\vec{E}$ implies that

$$\frac{\partial}{\partial r}\left(\frac{B}{r^2} - Ar\right)|_{r=R} \cos\theta = -\frac{P\cos\theta}{\epsilon_0}$$
$$\Rightarrow \quad \left(-\frac{2B}{R^3} - A\right) = -\frac{P}{\epsilon_0}$$

Meanwhile, the continuity of the potential across the boundary implies that

$$\begin{aligned} AR &= \frac{B}{R^2} \\ A &= \frac{B}{R^3} \end{aligned}$$

so

$$\begin{aligned}
\left(\frac{2B}{R^3}\right) &= \frac{P}{\epsilon_0} \\
\Rightarrow \quad \frac{3B}{R^3} &= \frac{P}{\epsilon_0} \\
\Rightarrow \quad B &= \frac{PR^3}{3\epsilon_0} \\
\Rightarrow \quad A &= \frac{P}{3\epsilon_0}
\end{aligned}$$

and

$$\begin{aligned}
V_{in} &= \frac{Pr}{3\epsilon_0}\cos\theta, \\
V_{out} &= \frac{PR^3}{3\epsilon_0 r^2}\cos\theta.
\end{aligned}$$

Inside the sphere,

$$\begin{aligned}
\vec{E}_{in} &= -\nabla V_{in} \\
&= -\frac{P}{3\epsilon_0}\left(\frac{\partial}{\partial r}(r\cos\theta)\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}(r\cos\theta)\hat{\theta}\right) \\
&= -\frac{P}{3\epsilon_0}(\cos\theta\hat{r} - \sin\theta\hat{\theta}) \\
&= -\frac{P\hat{z}}{3\epsilon_0} \\
&= -\frac{\vec{P}}{3\epsilon_0}
\end{aligned}$$

When the sphere is placed in the field in the external field $\vec{E}_0$, the resultant field

$$\vec{E} = \vec{E}_0 - \frac{1}{3\epsilon_0}\vec{P}$$

Now,

$$\begin{aligned}
\vec{D} &= \vec{P} + \epsilon_0\vec{E} \\
\Rightarrow \quad \epsilon_0\epsilon_r\vec{E} &= \vec{P} + \epsilon_0\vec{E} \\
\Rightarrow \quad \vec{P} &= \epsilon_0(\epsilon_r - 1)\vec{E}
\end{aligned}$$

Substituing inside,

$$\begin{aligned} \vec{E} &= \vec{E_0} - \frac{1}{3\epsilon_0}\vec{P} \\ \Rightarrow \quad \vec{E} &= \vec{E_0} - \frac{\epsilon_r - 1}{3}\vec{E} \\ \Rightarrow \quad \vec{E} &= \frac{3}{\epsilon_r + 2}\vec{E_0} \end{aligned}$$