PC3231 Electricity and Magnetism 2

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Suggested solutions for AY2007/08

Question 1

Part (A)
If we consider regions with no current, Ampere's law says:

$$\nabla \times \vec{B} = 0.$$

This allows us to write $\vec{B}$ as a gradient of some scalar function,

$$\vec{B} = \nabla V(\vec{r}, t).$$

For the magnetic vector potential, from the fact that there are no monopoles,

$$\nabla \cdot \vec{B} = 0 \quad \Rightarrow \quad \vec{B} = \nabla \times \vec{A}.$$

Choose $\nabla \cdot \vec{A} = 0$, we have $\nabla^2 \vec{A} = \mu_0 \vec{J}$, which are three Poisson's equations, one for each vector component.
For line currents, the solutin is analogous to the case of electrostatics: (note: $\vec{r} - \vec{r}' = \vec{\mathfrak{r}}$)

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{I}}{\mathfrak{r}} dl'$$

In this case, the scalar potential does not exist.

Part (B)
Gauss's Law in media: Given auxiliary fields: $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

$$\nabla \cdot \vec{D} = \rho_f \quad \rightarrow \quad \oint \vec{D} \cdot d\vec{a} = Q_{\mathrm{f,encl}}$$

$$\vec{D} = \frac{Q}{4\pi r^2} \hat{r}$$

Linear media, $\vec{D} = \epsilon \vec{E}$. And inside the metal sphere, $\vec{E} = \vec{P} = \vec{P} = 0$ (a conductor)

$$\begin{aligned} \vec{E} &= \begin{cases} \frac{Q}{4\pi\epsilon r^2} \hat{r} & \text{for} \quad a < r < b \\ \frac{Q}{4\pi\epsilon_0 r^2} \hat{r} & \text{for} \quad r > b \end{cases} \\ V &= -\int_{\infty}^{0} \vec{E} \cdot d\vec{l} = \int_{\infty}^{b} \frac{Q}{4\pi\epsilon_0 r^2} dr - \int_{a}^{b} \frac{Q}{4\pi\epsilon r^2} dr + 0 \\ &= \frac{Q}{4\pi} \left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b} \right), \quad V(\vec{r} \to \infty) = 0 \end{aligned}$$

At the inner surface, $r = a$,$\vec{n}$ directed *inside*, so

$$\sigma_b = \vec{P} \cdot \vec{n} = -\frac{\epsilon_0 \chi_e Q}{4\pi\epsilon a^2}$$

Outer surface, $r = b$, $\vec{n}$ directed *outside*,

$$\sigma_b = \vec{P} \cdot \vec{n} = +\frac{\epsilon_0 \chi_e Q}{4\pi\epsilon b^2}$$

Total energy:

$$U = \frac{1}{2}\epsilon_0 \int E^2 d\tau$$

$$E^2 = \begin{cases} \left(\frac{Q}{4\pi\epsilon}\right)^2 \frac{1}{r^4}, & a < r < b \\ \left(\frac{Q}{4\pi\epsilon_0}\right)^2 \frac{1}{r^4}, & r > b \end{cases}$$

$$\begin{aligned} U_1 &= \frac{1}{2}\epsilon_0 \int \left(\frac{Q}{4\pi\epsilon}\right)^2 \frac{1}{r^4} r^2 \sin\theta dr d\phi d\theta \\ &= \frac{1}{2}\epsilon_0 \frac{Q^2}{4\pi^2\epsilon^2} \underbrace{\int_0^{2\pi} d\phi}_{s} \underbrace{\int_0^{\pi} \sin\theta}_{2} \underbrace{\int_a^b \frac{1}{r^2} dr}_{a^{-3}-b^{-3}} \\ &= \frac{1}{2}\epsilon_0 \frac{Q^2}{\pi\epsilon^2}\left(a^{-3} - b^{-3}\right) \end{aligned}$$

Similarly, integrating the electric field for the region outside,

$$U_2 = \frac{1}{2}\epsilon_0 \frac{Q^2}{\pi\epsilon_0^2} \frac{1}{b}$$

So the total energy:

$$U = \frac{1}{2}\epsilon_0 \left[\frac{Q^2}{\pi\epsilon_0^2 b} + \frac{Q^2}{\pi\epsilon^2}\left(\frac{1}{a^3} - \frac{1}{b^3}\right)\right]$$

Question 2

Part (A)
TE_{00} mode, $E_z = 0$, from Gauss' Law:

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0.$$

$$\begin{aligned} \text{Also, no monopoles: } \nabla \cdot \vec{B} &= 0 \\ \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} &= 0 \end{aligned}$$

Faraday's Law,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

But curl of $\vec{E}$ is zero, this implies that $\partial \vec{B}/\partial t = 0$, $\vec{B}$ is constant.
Boundary conditions, $\vec{B} = 0$ at the boundaries, so B_z is constant and zero - no field.

Part (B)

$$\begin{aligned} B_z &= B_0 \cos\frac{\pi x}{a} \cos\frac{\pi y}{b} \\ \frac{\partial B_z}{\partial y} &= -B_0 \cos\frac{\pi x}{a} \sin\frac{\pi y}{b} \left(\frac{\pi}{b}\right) \\ & \longrightarrow +\omega \frac{\partial B_z}{\partial y} = -B_0 \left(\frac{\pi}{b} \cos\frac{\pi x}{a} \sin\frac{\pi y}{b}\right) \\ \frac{\partial B_z}{\partial x} &= -B_0 \sin\frac{\pi x}{a} \cos\frac{\pi y}{b} \left(\frac{\pi}{a}\right) \\ & \longrightarrow -\omega \frac{\partial B_z}{\partial x} = +B_0 \left(\frac{\pi}{a}\right) B_0 \sin\frac{\pi x}{a} \cos\frac{\pi y}{b} \end{aligned}$$

Substitute these expressions into the given formulas for E_x and E_y,

$$\begin{aligned} E_x &= -\frac{i}{\omega^2/c^2 - k^2} \left(\frac{\pi}{b} B_0 \cos\frac{\pi x}{a} \sin\frac{\pi y}{b}\right), \\ E_y &= +\frac{i}{\omega^2/c^2 - k^2} \left(\frac{\pi}{a} B_0 \cos\frac{\pi x}{a} \sin\frac{\pi y}{b}\right), \\ \omega &= c\pi \sqrt{\frac{1}{a^2} + \frac{1}{b^2}} \end{aligned}$$

Question 3

Part (A)

Consider an object of length L moving at velocity v: The observer sees the object as having the length L'. The time it takes for light from the back of the object to arrive at the observer is

$$t = \frac{L' \cos\theta}{c} = \frac{\hat{\mathfrak{r}} \cdot \vec{v}}{c}$$

At the sametime the object has moved a distance $L' - L$, so,

$$\frac{\hat{\mathfrak{r}} \cdot \vec{v}}{c} = \frac{L' - L}{v}, \quad \text{or,} \quad L' = \frac{L}{1 - \hat{\mathfrak{r}} \cdot \vec{v}/c}$$

Distances perpendicular to the motion is not affected, so the apparent volume measured by the observer is:

$$\tau' = \frac{\tau}{1 - \hat{\mathfrak{r}} \cdot \vec{v}/c}$$

Since this formula does not take into account the size of the object, this applies equally well to point particles. SO the potential introduces a factor of

$$\frac{1}{1 - \hat{\mathfrak{r}} \cdot \vec{v}/c}$$

Part (B)
Based on the hints, the horizontal components can be shown to cancel by symmetry about the y-axis. So only need to consider the y-components.

$$\begin{aligned}
\vec{E} &= \frac{q}{4\pi\epsilon_0} \frac{1 - \frac{v^2}{c^2}}{\left(1 - \frac{v^2 \sin^2\theta}{c^2}\right)^{3/2}} \frac{\hat{R}}{R^2} \sin\theta \hat{y} \\
&= \frac{q}{4\pi\epsilon_0} \frac{(1-a^2)\sin\theta}{1 - a^2 \sin^2\theta} \frac{\hat{R}}{R^2} \hat{y}, \quad a = \frac{v}{c} \qquad (1)
\end{aligned}$$

Note that

$$x = d\cot\theta, \quad \longrightarrow \quad dx = -d \text{ cosec}^2\theta d\theta$$

Therefore

$$\frac{q}{dx} = \lambda = -\frac{q}{d \text{ cosec}^2\theta d\theta}. \qquad (2)$$

The small change in electric field:

$$\begin{aligned}
d\vec{E} &= \frac{1}{4\pi\epsilon_0 d} \frac{q d\theta}{d \text{ cosec}^2\theta d\theta} \frac{(1-a^2)\sin\theta}{(1 - a^2\sin^2\theta)^{3/2}} \hat{y} \\
&= \frac{1}{4\pi\epsilon_0 d} \frac{\lambda}{d} \frac{(1-a^2)\hat{y}}{(1 - a^2\sin^2\theta)^{3/2}} d\theta \\
\vec{E} &= \int_0^\pi \frac{1}{4\pi\epsilon_0} \frac{\lambda}{d} \frac{(1-a^2)\hat{y}}{(1 - a^2\sin^2\theta)^{3/2}} d\theta
\end{aligned}$$

Change variables, $z = \cos\theta$, limitss $[0, \pi] \to [-1, 1]$, $dz = -\sin\theta d\theta$

$$\begin{aligned}
\vec{E} &= -\int_1^{-1} \frac{1}{4\pi\epsilon_0 d} \frac{\lambda}{d} \frac{(1-a^2)dz)}{(1 - a^2 + a^2 z^2)^{3/2}} \\
&= \frac{1}{4\pi\epsilon_0 d} \frac{\lambda}{d} (1-a^2) \int_{-1}^{1} \frac{dz}{(1 - a^2 + a^2 z^2)^{3/2}} \\
&= \frac{1}{4\pi\epsilon_0 d} \frac{\lambda}{d} (1-a^2) \left[\frac{z}{a^3 (a^{-2} - 1)\sqrt{a^{-2} - 1 + z^2}} \right]_{-1}^{1} \\
&= \frac{1}{4\pi\epsilon_0 d} \frac{\lambda}{d} (1-a^2) \frac{1}{a^2 (a^{-2} - 1)} \\
&= \frac{1}{2\pi\epsilon_0 d} \frac{\lambda}{d}
\end{aligned}$$

Question 4

Part (i)

Retarded time,

$$
\begin{aligned}
t_r = t - \frac{\mathfrak{r}}{c} &= t - \sqrt{\frac{x^2 + r^2}{c}} \\
&= t - \frac{1}{c}\left(\sqrt{r^2 + x^2} + x - x\right) \\
&= t = \frac{x}{c} - \frac{1}{c}\left(\sqrt{r^2 + x^2} - x\right) \\
&= t - \frac{x}{c} - u \\
du &= \frac{1}{c}\frac{r}{\sqrt{r^2 + x^2}}dr
\end{aligned}
$$

The vector potential,

$$
\begin{aligned}
\vec{A} &= \frac{\mu_0}{4\pi}\int \frac{\vec{K}}{\sqrt{x^2 + r^2}} 2\pi r dr \\
&= \frac{\mu_0}{2}\int \vec{K}(t_r)\frac{rdr}{sqrtr^2 + x^2} \\
&= \frac{\mu_0 c}{2}\hat{z}\int K(t - \frac{x}{c} - u)du
\end{aligned}
$$

Part (ii)

No charge,

$$
\begin{aligned}
\vec{E} &= -\frac{\partial \vec{A}}{\partial t} \\
&= -\frac{\mu_0 c}{2}\hat{z}\int \dot{K}(t - \frac{x}{c} - u)du
\end{aligned}
$$

Magnetic field,

$$
\vec{B} = \nabla \times \vec{A}
$$