PC 3231 - Electricity and Magnetism 2

AY04/05 SEM 1
Suggested Solutions

Q1

A

$$\rho(r',\theta') = \frac{kR}{r'^2}(R-2r')\sin\theta'$$

$$\begin{aligned} V_{mono} &= \frac{1}{4\pi\epsilon_0}(\frac{1}{r}\int \rho(\tau')\ d\tau') \\ &= \frac{2\pi}{4\pi\epsilon_0}\frac{1}{r}\int_0^R \frac{kR}{r'^2}(R-2r')r'^2\ dr' \int_0^\pi \sin\theta\sin\theta\ d\theta \end{aligned}$$

Consider the dr' integration :

$$\begin{aligned} \int_0^R (R-2r')\ dr' &= Rr' - r'^2\ |_0^R \\ &= 0 \end{aligned}$$

so the monoopole term is zero.

$$\begin{aligned} V_{dipole} &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\theta'\rho\ d\tau' \\ &= \frac{kR}{2\epsilon_0 r^2}\int_0^R r'(R-2r')\int_0^\pi \sin^2\theta'\cos\theta'\ d\theta' \end{aligned}$$

Consider the $d\theta'$ integration:

$$\begin{aligned} \int_0^\pi \sin^2\theta'\cos\theta' d\theta' &= \frac{1}{3}\sin^3\theta|_0^\pi \\ &= 0 \end{aligned}$$

so the dipole term is zero.

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\begin{aligned}
V_{quad} &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int r'^2(\frac{3}{2}\cos\theta - \frac{1}{2})\rho\ d\tau' \\
&= \frac{kR}{4\pi\epsilon_0 r^3}\int_0^R (R-2r')r'^2\ dr'\int_0^\pi (3\cos^2\theta - 1)\sin\theta'\ d\theta' \\
&= \frac{kR}{4\epsilon_0 r^3}(\frac{Rr'^3}{3} - \frac{1}{2}r'^4)|_0^R(\cos\theta - \cos^3\theta)|_0^\pi \\
&= \frac{kR}{4\epsilon_0 r^3}(-\frac{R^4}{6})(-2-2) \\
&= \frac{kR^5}{6\epsilon_0 r^3}
\end{aligned}
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(B)

(i)

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\begin{aligned}
\vec{E}_{in} &= -\nabla V_{in} \\
&= -\frac{\rho}{3\epsilon_0}(\frac{\partial}{\partial r}(r\cos\theta)\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}(r\cos\theta)\hat{\theta} \\
&= -\frac{\rho}{3\epsilon_0}(\cos\theta\hat{r} - \sin\theta\hat{\theta}) \\
&= -\frac{\rho}{3\epsilon_0}\hat{z}
\end{aligned}
$$

(ii)

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\begin{aligned}
\vec{E}_{out} &= -\nabla V_{out} \\
&= -\frac{\rho R^3}{3\epsilon_0}(\frac{\partial}{\partial r}\frac{\cos\theta}{r^2}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\frac{\cos\theta}{r^2}\hat{\theta}) \\
&= -\frac{\rho R^3}{3\epsilon_0}(-\frac{2\cos\theta}{r^3}\hat{r} - \frac{\sin\theta}{r^3}\hat{\theta}) \\
&= \frac{\rho R^3}{3\epsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})
\end{aligned}
$$

Q2

(A)

The magnetic field $\vec{B}$ inside a soleniod is $\mu_0 n I \hat{z}$, and 0 outside. For $\vec{A}$ inside the solenoid, consider a circular Amperian loop of radius $s < R$ coaxial with the wire. Then,

$$\oint \vec{A} \cdot d\vec{l} = \int \vec{B} \cdot d\vec{a}$$
$$: \;\; A(2\pi s) = \int_0^s (\mu_0 n I)(2\pi s \; ds)$$
$$: \;\; A = \frac{1}{2\pi s}(\pi \mu_0 n I s^2)$$
$$: \;\; \vec{A} = \frac{\mu_0 n I s}{2}\hat{\phi}$$

For $\vec{A}$ outside the solenoid, consider the same Amperian loop but with $s > R$. Then,

$$\oint \vec{A} \cdot d\vec{l} = \int \vec{B} \cdot d\vec{a}$$
$$: \;\; A(2\pi s) = \int_0^R (\mu_0 n I)(2\pi s \; ds)$$
$$: \;\; A = \frac{1}{2\pi s}(\pi \mu_0 n I R^2)$$
$$: \;\; \vec{A} = \frac{\mu_0 n I R^2}{2s}\hat{\phi}$$

$$\begin{aligned} \nabla \times \vec{A}_{in} &= \frac{\mu_0 n I}{2}(\nabla \times s)\hat{\phi} \\ &= \frac{\mu_0 n I}{2}\frac{1}{s}\frac{\partial}{\partial s}s^2\hat{z} \\ &= \mu_0 n I \hat{z} \\ &= \vec{B}_{in} \end{aligned}$$

$$\nabla \times \vec{A}_{out} = 0 = \vec{B}_{out}$$

B

Consider 2 sets of potential:

$$\begin{aligned} \vec{A}' &= \vec{A} + \vec{\alpha} \\ V' &= V + \beta \end{aligned}$$

such that $\vec{A}$ and $\vec{A}'$ give the same $\vec{B}$ and $\vec{E}$:

$$\vec{B} = \nabla \times \vec{A} = \nabla \times \vec{A}'$$
$$\nabla \times \vec{\alpha} = 0.$$

Writing α as the gradient of a scalar λ,

$$\begin{aligned}
\alpha &= \nabla\lambda \ (\nabla \times \nabla\lambda = 0) \\
\vec{E} &= -\nabla V - \frac{\partial}{\partial t}\vec{A} \\
&= -\nabla V' - \frac{\partial}{\partial t}\vec{A}'
\end{aligned}$$

hence

$$\nabla\beta + \frac{\partial}{\partial t}\vec{\alpha} = 0$$
$$\nabla(\beta + \frac{\partial}{\partial t}\lambda) = 0$$

The term in parantheses is independent of position, but it could depend on time :

$$\beta = -\frac{\partial}{\partial t} + k(t)$$

We can absorb $k(t)$ into λ, without affecting the gradient. Hence,

$$\vec{A}' = \vec{A} + \nabla\lambda,$$
$$V' = V - \frac{\partial}{\partial t}\lambda.$$

We can add $\nabla\lambda$ to $\vec{A}$, provided we simulatenously subtract $\vec{\partial}\partial t\lambda$ from V.

Q3

(A)

As $\vec{A}_{above} = \vec{A}_{below}$ at every point on the surface, $\frac{\partial \vec{A}}{\partial x}$ and $\frac{\partial \vec{A}}{y}$ are continuous, and the discontinuity is confined to $\frac{\partial \vec{A}}{\partial z}$.

$$\vec{B} = \nabla \times \vec{A}$$

$$\begin{aligned}
\vec{B}_{above} - \vec{B}_{below} &= \hat{x}\left(-\frac{\partial A_{y(above)}}{\partial z} - \frac{\partial A_{y(below)}}{\partial z}\right) + \hat{y}\left(\frac{\partial A_{x(above)}}{\partial z} - \frac{\partial A_{x(below)}}{z}\right) \\
&= \mu_0(\vec{K} \times \hat{n}) \\
&= -\mu_0 K \hat{y}
\end{aligned}$$

Equating $\hat{x}$ and $\hat{y}$ components,
y component :

$$\left(\frac{\partial A_y}{\partial z}\right)_{(above)} = \left(\frac{\partial A_y}{\partial z}\right)_{below}$$

x component :

$$\left(\frac{\partial A_x}{\partial z}\right)_{(above)} - \left(\frac{\partial A_x}{\partial z}\right)_{(below)} = -\mu_0 k$$

The normal derivative of the component of $\vec{A}$ parallel to $\vec{k}$ suffers a discontinuity

$$\frac{\partial \vec{A}_{above}}{\partial n} - \frac{\vec{A}_{below}}{\partial n} = -\mu_0 \vec{k}$$

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(B)

For the static case the magnetic field due to a sheet current is $\vec{E} = \frac{\mu_0 K}{2}\hat{y}$. It is independent of the distance x from the plane. The magnetic vector potential is then given by $\vec{A} = \frac{\mu_0 K}{2} x\hat{z}$. Thus the retarded potential is expected to be in the z direction. For above the plane, in the region $\pm x$, we expect therefore

$$\begin{aligned}
\vec{A} &= \frac{\mu_0}{4\pi}\int \frac{\vec{K}}{r}\, da \\
&= \frac{\mu_0 \hat{z}}{4\pi}\int \frac{K(t_r)}{\sqrt{r^2+x^2}} 2\pi r\ dr \\
&= \frac{\mu_0 \hat{z}}{2}\int \frac{K(t-\sqrt{r^2+x^2}c}{\sqrt{r^2+x^2}} r\ dr
\end{aligned}$$

The maximum r is given by $t - \sqrt{r^2 + x^2}/c = 0$, or $r_{max} = \sqrt{c^2t^2 - x^2}$. (since $K(t) = 0$ for $t < 0$).

$$\begin{aligned}
\vec{A} &= \frac{\mu_0 K_0 \hat{z}}{2} \int_0^{\infty} \frac{r}{\sqrt{r^2 + x^2}}\, dr \\
&= \frac{\mu_0 K_0 \hat{z}}{2} \sqrt{r^2 + x^2}|_0^{r\infty} \\
&= \frac{\mu_0 K_0 (ct - x)}{2} \hat{z}
\end{aligned}$$

$$\begin{aligned}
\vec{E}(x,t) &= \frac{\partial \vec{A}}{\partial t} \\
&= -\frac{\mu_0 K_0 c}{2} \hat{z}
\end{aligned}$$

for $ct > x$, and 0, for $ct < x$.

$$\begin{aligned}
\vec{B}(x,t) &= \nabla \times \vec{A} \\
&= \frac{\mu_0 K_0 \hat{y}}{2}
\end{aligned}$$

for $ct > x$, and 0, for $ct < x$.

Q4

(A)

(i)

The plane wave solution for $\vec{E}$ is

$$\vec{E}(z,t) = \vec{E}_0 \mathrm{e}^{i(kz-\omega t)}$$

Substituting inside

$$\nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t},$$

gives

$$k^2 = -\mu\epsilon\omega^2 - i\mu\sigma\omega$$

. Write $k = k_+ + ik_-$. The above becomes

$$k_+^2 - k_-^2 + 2ik_-k_+ = \mu\epsilon\omega_+^2 i\mu\sigma\omega$$

. Comparing real and imaginery parts gives

$$\begin{aligned} k_+^2 - k_-^2 &= \mu\epsilon\omega^2 \\ 2k_-k_+ = \mu\sigma\omega \end{aligned}$$

$$\begin{aligned}
k_- &= \frac{\mu\sigma\omega}{2k_+} \\
k_+^2 - \left(\frac{\mu\sigma\omega}{2k_+}\right)^2 &= \mu\epsilon\omega^2 \\
k_+^4 - k_+^2\mu\epsilon\omega^2 - \left(\frac{\mu\sigma\omega}{2}\right)^2 &= 0 \\
k_+^2 &= \frac{\mu\epsilon\omega^2 \pm \sqrt{(\mu\epsilon\omega^2)^2 + 4\left(\frac{\mu\sigma\omega}{2}\right)^2}}{2} \\
&= \frac{\mu\epsilon\omega^2}{2} \pm \frac{\mu\epsilon\omega^2}{2}\sqrt{1 + \frac{(\mu\sigma\omega)^2}{(\mu\epsilon\omega^2)^2}} \\
&= \frac{\mu\epsilon\omega^2}{2}\left[1 \pm \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}\right] \\
k_+ &= \omega\sqrt{\frac{\mu\epsilon}{2}}\left[1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}\right]^{\frac{1}{2}}
\end{aligned}$$

A similar derivation for k_- results in

$$k_\pm = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} \pm 1\right]^{\frac{1}{2}}$$

(ii)

Consider $\vec{E}$ polarised in the x-direction.

$$\begin{aligned}\vec{E}(z,t) &= \mathbf{E}_0 \mathrm{e}^{-k_- z}\mathrm{e}^{i(k_+ z-\omega t)}\hat{x} \\ \vec{B}(z,t) &= \frac{|k|}{\omega}\mathbf{E}_0 \mathrm{e}^{-k_- z}\mathrm{e}^{i(k_+ z-\omega t)}\hat{y}\end{aligned}$$

Now,

$$\begin{aligned}k &= k_+ + ik_- = |k|\mathrm{e}^{i\phi} = |k|(\cos\phi + i\sin\phi) \\ |k| &= \sqrt{k_+^2 + k_-^2} = \omega\sqrt{\epsilon\mu(1+\left(\frac{\sigma}{\omega\epsilon}\right)^2)^{\frac{1}{2}}}\end{aligned}$$

$$\begin{aligned}k_+ &= |k|\cos\phi \\ k_- &= |k|\sin\phi \\ \phi &= \tan^{-1}\frac{k_-}{k_+}\end{aligned}$$

The amplitudes are complex in general,

$$\begin{aligned}\mathbf{E}_0 &= E_0\mathrm{e}^{i\delta_E} \\ \mathbf{B}_0 &= B_0\mathrm{e}^{i\delta_B} \\ B_0\mathrm{e}^{i\delta_B} &= \frac{|k|\mathrm{e}^{i\phi}}{\omega}E_0\mathrm{e}^{i\delta_E}\end{aligned}$$

so $\vec{E}$ and $\vec{B}$ are no longer in phase, and

$$\delta_B - \delta_E = \phi = \tan^{-1}\frac{k_-}{k_+}$$

so the magnetic field lags behind the electric field $\vec{E}$.

In a good conductor,

$$\begin{aligned}k_+ &= \omega\sqrt{\frac{\mu\epsilon}{2}}\left[1+\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}\right]^{\frac{1}{2}} \\ &\approx \omega\sqrt{\frac{\mu\epsilon}{2}}\left(\frac{\sigma}{\epsilon\omega}\right)^{\frac{1}{2}} \\ &\approx k_-\end{aligned}$$

so

$$\begin{aligned}\tan^{-1}\frac{k_-}{k_+} &\approx \tan^{-1}1 \\ &= \frac{\pi}{4}\end{aligned}$$

and the magnetic field lags behind the electric field by 45^0.

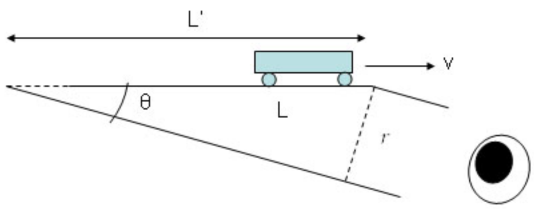
iii

$$\begin{aligned} < u > &= \frac{1}{2} < (\epsilon E^2 + \frac{1}{\mu} B^2) > \\ &= \frac{1}{4} E_0^2 \mathrm{e}^{-2k_- z} (\epsilon + \frac{k^2}{\mu\omega^2}) \\ &= \frac{1}{4} E_0^2 \mathrm{e}^{-2k_- z} \epsilon \left(1 + \sqrt{1 + (\frac{\sigma}{\epsilon\omega})^2} \right) \end{aligned}$$

The 1 witin the paranthesis comes from the electrical contribution and the other term, which is larger than 1, comes from the magnetic contribution. Hence, the magnetic contribution always dominates.

(B)

Consider the geometry as shown : In general, if the velocity of the train $\vec{v}$ makes



an angle θ with the observer's line of sight, the extra distance covered is $L' \cos\theta$. In the time $L' \cos\theta/c$, the train moves a distance $(L' - L)$, such that

$$\frac{L' \cos\theta}{c} = \frac{L' - L}{v}$$

. That is,

$$L' = \frac{L}{1 - \frac{v \cos\theta}{c}}$$

, so that the effect of retardation is the multiplication of the volume of a point charge by a factor of

$$\frac{1}{1 - \frac{\mathrm{r} \cdot \vec{v}}{c}}$$

, so that

$$\int \rho(\vec{r'}, t_r)\ d\tau' = \frac{q}{1 - \frac{\vec{\mathrm{r}} \cdot \vec{v}}{c}}$$

The electric potential is thus

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{\mathrm{r}c - \vec{\mathrm{r}} \cdot \vec{v}}$$

. The current density is $\rho\vec{v}$.

$$\begin{aligned}
\vec{A}(\vec{r},t) &= \frac{\mu_0}{4\pi}\int \frac{\rho(\vec{r}',t_r)\vec{v}(t_r)}{\mathrm{r}}\ d\tau' \\
&= \frac{\mu_0}{4\pi}\frac{\vec{v}}{\mathrm{r}}\int \rho(\vec{r}',t_r)\ d\tau' \\
&= \frac{\mu_0}{4\pi}\frac{qc\vec{v}}{\mathrm{r}c-\vec{\mathrm{r}}\cdot\vec{v}} \\
&= \frac{\vec{v}}{c^2}V(\vec{r},t).
\end{aligned}$$