PC 3231 - Electricity and Magnetism 2

AY03/04 SEM 1
Suggested Solutions

Q1

a

$$\vec{B} = \vec{B}_{straight} + \vec{B}_{bent}$$

Both $\vec{B}_{straight}$ and $\vec{B}_{bent}$ point upward and are perpendicular to the plane of the loop.

$$\begin{aligned} B_{straight} &= \frac{\mu_0 I}{4\pi R}(2\sin\theta) \\ &= \frac{\mu_0 I}{2\pi R}\sin\theta. \end{aligned}$$

For $\vec{B}_{bent}$, consider an current element dl.

$$dB_{bent} = \frac{\mu_0 I dl}{4\pi R^2} = \frac{\mu_0 I}{4\pi R}\, d\alpha$$

where α is the angle subtended by dl.

$$B_{bent} = \frac{\mu_0 I}{4\pi R}\int_0^{2\pi-2\theta} d\alpha = \frac{\mu_0 I}{2\pi R}(\pi - \theta)$$

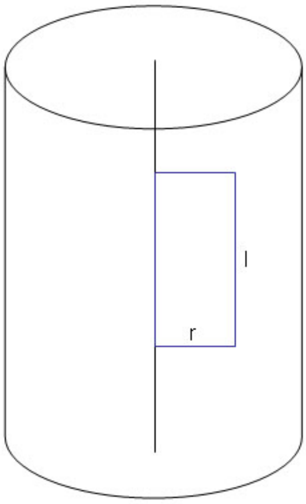
The total field is

$$\begin{aligned} B &= B_{straight} + B_{bent} \\ &= \frac{\mu_0 I}{2\pi R}(\pi - \theta + \sin\theta) \end{aligned}$$

b

$$\begin{aligned} & \vec{B} = \nabla \times \vec{A} \\ \Rightarrow \quad & \int \vec{B}\cdot\, d\vec{a} = \int \nabla \times \vec{A}\cdot\, d\vec{a} = \int \vec{A}\cdot\, d\vec{l} \end{aligned}$$

Set $\vec{A}$ to be 0 in the middle of the wire. Then, drawing a rectiangular Amperian loop with its normal in the $\hat{s}$ direction with one edge of length l in the middle of the wire and the other edge at $r < R$ away as shown, we have



$$
\begin{aligned}
lA &= \int \vec{B} \cdot da \\
&= \int_0^r \frac{\mu_0 I r}{2\pi R^2} l \; dr \\
&= \frac{\mu_0 I l}{4\pi R^2} r^2
\end{aligned}
$$

so for $r < R$, $\vec{A} = \frac{\mu_0 I}{4\pi R^2} r^2 \; \hat{z}$.

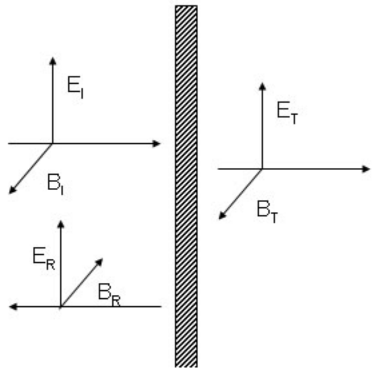
For $\vec{A}$ outside the wire, consider a rectangular Amperian loop with one end at $r = R$ and the opposite edge at $r > R$. Then,

$$
\begin{aligned}
l(A - \frac{\mu_0 I}{4\pi R^2} R^2) &= \int_R^r \frac{\mu_0 I}{2\pi r} l \; dr \\
\Rightarrow \vec{A} &= \frac{\mu_0 I}{2\pi} (\ln(\frac{r}{R}) + \frac{1}{2}) \hat{z}.
\end{aligned}
$$

Q2

Consider the following geometry :

a



Continuity of the component of $\vec{E}$ perpendicular to the boundary requires that

$$E_I + E_R = E_T$$

The continuity of the componenet of $\vec{H}$ parallel to the boundary requires that

$$\begin{aligned} \frac{1}{\mu_1}(B_I - B_R) &= \frac{1}{\mu_2}B_T \\ \Rightarrow \quad \frac{1}{\mu_1 v_1}(E_I - E_R) &= \frac{1}{\mu_1 v_2}E_T. \end{aligned}$$

Assuming $\mu_1 = \mu_2 = \mu_0$, the above simplifies to

$$n_1(E_I - E_R) = n_2 E_T.$$

Solving for E_I and E_R gives

$$\begin{aligned} E_R &= \frac{n_1 - n_2}{n_1 + n_2}, \\ E_I &= \frac{2n_1}{n_1 + n_2}. \end{aligned}$$

Thus,

$$
\begin{aligned}
R &= \left(\frac{E_R}{E_I}\right)^2 \\
&= \left(\frac{n_1 + n_2}{n_1 + n_2}\right)^2, \\
T &= \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_T}{E_I}\right)^2 \\
&= \frac{n_2}{n_1}\left(\frac{2n_1}{n_1 + n_2}\right)^2 \\
&= \frac{4n_1 n_2}{(n_1 + n_2)^2}.
\end{aligned}
$$

b

Let the electric and magnetic fields be

$$
\begin{aligned}
E &= E_0 \mathrm{e}^{-k_- z} \cos(k_+ z - \omega t + \delta_e)\hat{x} \\
B &= \frac{|k|}{\omega} E_0 \mathrm{e}^{-k_- z} \cos(k_+ z - \omega t + \delta_b)\hat{y}
\end{aligned}
$$

The time averaged energy density is then

$$
\begin{aligned}
u &= \frac{1}{2} < (\epsilon E^2 + \frac{1}{\mu} B^2) > \\
&= \frac{1}{4} E_0^2 \mathrm{e}^{-2k_- z} (\epsilon + \frac{1}{\mu}\left(\frac{|k|}{\omega}\right)^2)
\end{aligned}
$$

as the time average of the $\cos^2$ terms is $\frac{1}{2}$.

Consider monochrmoatic wave incident on a thick slab of thickness Δz and cross sectional area A.

Average flow into slab is $< S > A$, i.e.

$$
\frac{1}{2}\frac{k_+}{\mu\omega} E_0^2 \mathrm{e}^{-2k_- z} A
$$

at point z. The average flow out at $z' = z + \Delta z$ is evaluated by the same formula at z'.

The power disspated is the difference between "in" and "out"

$$\begin{aligned} < P > &= \frac{1}{2}\frac{k_+}{\mu\omega}E_0^2 A[-\frac{d}{dz}\mathrm{e}^{-2k_- z}\Delta z] \\ &= \frac{k_+ k_-}{\mu\omega}E_0^2 \mathrm{e}^{-2k_- z}(A\Delta z) \\ &= \frac{1}{2}\sigma E_0^2 \mathrm{e}^{-2k_- z} A\Delta z. \end{aligned}$$

Q3

a

i

Consider 2 sets of potential:

$$\begin{aligned} \vec{A}' &= \vec{A} + \vec{\alpha} \\ V' &= V + \beta \end{aligned}$$

such that $\vec{A}$ and $\vec{A}'$ give the same $\vec{B}$ and $\vec{E}$:

$$\vec{B} = \nabla \times \vec{A} = \nabla \times \vec{A}'$$
$$\nabla \times \vec{\alpha} = 0.$$

Writing α as the gradient of a scalar λ,

$$\begin{aligned} \alpha &= \nabla\lambda \ (\nabla \times \nabla\lambda = 0) \\ \vec{E} &= -\nabla V - \frac{\partial}{\partial t}\vec{A} \\ &= -\nabla V' - \frac{\partial}{\partial t}\vec{A}' \end{aligned}$$

hence

$$\nabla\beta + \frac{\partial}{\partial t}\vec{\alpha} = 0$$
$$\nabla(\beta + \frac{\partial}{\partial t}\lambda) = 0$$

The term in parantheses is independent of position, but it could depend on time :

$$\beta = -\frac{\partial}{\partial t} + k(t)$$

We can absorb $k(t)$ into λ, without affecting the gradient. Hence,

$$\vec{A}' = \vec{A} + \nabla\lambda,$$
$$V' = V - \frac{\partial}{\partial t}\lambda.$$

We can add $\nabla\lambda$ to $\vec{A}$, provided we simulatenously subtract $\vec{\partial}\partial t\lambda$ from V.

ii

$$\begin{aligned}\vec{E} &= -\nabla V - \frac{\partial}{\partial t}\vec{A} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}\\ \vec{B} &= \nabla\times\vec{A} = 0\end{aligned}$$

This is a set of potentials for a stationary point charge q at the origin, more usually

$$\begin{aligned}V &= \frac{1}{4\pi\epsilon_0}\frac{q}{r}\\ \vec{A} &= 0.\end{aligned}$$

Gauge transforming by λ, we have

$$\begin{aligned}V' &= V - \frac{\partial}{\partial t}\lambda = 0 - \left(-\frac{1}{4\pi\epsilon_0}\frac{q}{r}\right) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}\\ \vec{A}' &= \vec{A} + \nabla\lambda = -\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{r} + \frac{-1}{4\pi\epsilon_0}qt\frac{-1}{r^2}\hat{r} = 0\end{aligned}$$

as in 'usual' potentials of a point charge.

b

$$\vec{A}(r,t) = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty}\frac{q_0\delta(t-s/c)}{s}\,dz$$

but $s = \sqrt{r^2+z^2}$, so the integrand is even in z :

$$\vec{A}(r,t) = \frac{\mu_0 q_0}{4\pi}2\int_0^{\infty}\frac{\delta(t-s/c)}{s}\,dz$$

Now $z = \sqrt{s^2-r^2}$, implying that

$$dz = \frac{1}{2}\frac{2s\ ds}{\sqrt{s^2-r^2}} = \frac{s\ ds}{\sqrt{s^2-r^2}}$$

where $z = 0 \Rightarrow s = r$,$z = \infty \Rightarrow r = \infty$. Hence

$$\vec{A} = \frac{\mu_0 q_0}{2\pi}\int_r^{\infty}\frac{\delta(t-s/c)}{s}\frac{s\ ds}{\sqrt{s^2-r^2}}.$$

Now $\delta(t-s/c) = c\delta(s-ct)$, therefore

$$\vec{A} = \frac{\mu_0 q_0}{2\pi}\hat{z}c\int_r^{\infty}\frac{\delta(s-ct)}{\sqrt{s^2-r^2}}\,ds$$

or

$$\vec{A} = \frac{\mu_0 q_0}{2\pi} \frac{1}{\sqrt{(ct)^2 - r^2}} \hat{z}$$

(or zero, if $ct < r$).

$$E = -\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 q_0}{2\pi} \left(-\frac{1}{2}\right) \frac{2c^2 t}{[(ct)^2 - r^2]^{\frac{3}{2}}} \hat{z} = \frac{\mu_0 q_0 c^3 t}{2\pi[(ct)^2 - r^2]^{\frac{3}{2}}} \hat{z}$$

(or zero, if $t < r/c$).

$$\begin{aligned} \vec{B} = \nabla \times \vec{A} &= -\frac{\partial \vec{A}_z}{\partial t} \hat{\phi} \\ &= -\frac{\mu_0 q_0 c}{2\pi} \left(-\frac{1}{2}\right) \frac{-2r}{[(ct)^2 - r^2]^{\frac{3}{2}}} \hat{\phi} \\ &= -\frac{\mu_0 q_0 c r}{2\pi[(ct)^2 - r^2]^{\frac{3}{2}}} \hat{\phi}. \end{aligned}$$

(or zero, if $t < r/c$).

Q4

(i)

$$\begin{aligned}
\nabla V &= \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} \\
&= -\frac{p_0\omega}{4\pi\epsilon_0 c}\left\{\cos\theta[-\frac{1}{r^2}\sin\omega(t-r/c) - \frac{\omega}{rc}\cos\omega(t-r/c)] - \frac{\sin\theta}{r^2}\sin\omega(t-r/c)\hat{\theta}\right\} \\
&\approx \frac{p_0\omega^2}{4\pi\epsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\cos\omega(t-r/c)\hat{r},
\end{aligned}$$

$$\frac{\partial}{\partial t}\vec{A} = -\frac{\mu_0 p_0 \omega^2}{4\pi r}\cos[\omega(t-r/c)](\cos\theta\hat{r} - \sin\theta\hat{\theta}),$$

so

$$\begin{aligned}
\vec{E} &= -\nabla V - \frac{\partial \vec{A}}{\partial t} \\
&= -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\omega(t-r/c)\hat{\theta}.
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
\vec{B} &= \nabla \times \vec{A} \\
&= \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{\phi} \\
&= \frac{-\mu_0 q_0 \omega}{4\pi r}\left\{\frac{\omega}{c}\sin\theta\cos\omega(t-r/c) + \frac{\sin\theta}{r}\sin\omega(t-r/c)]\right\}\hat{\phi} \\
&\approx -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-\frac{r}{c}]\hat{\phi}.
\end{aligned}$$

(ii)

$$\begin{aligned}
< \vec{S} > &= \frac{1}{\mu_0}(< \vec{E} \times \vec{B} >) \\
&= \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right) < \cos\omega(t-r/c) >\right\}^2 \hat{r} \\
&= \left(\frac{\mu_0 p_0^4 \omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\hat{r}.
\end{aligned}$$

(iv)

$$\begin{aligned} < P > &= \int < \vec{S} > \cdot d\vec{a} \\ &= \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \int \frac{\sin^2\theta}{r^2} r^2 \sin\theta d\theta d\phi \\ &= \frac{\mu_0 p_0^2 \omega^4}{12\pi c} \end{aligned}$$

(v)

$$P = I^2 R = q_0^2 \omega^2 \sin^2 \omega t R$$

Average power,

$$< P >= \frac{1}{2} q_0^2 \omega^2 R$$

Equating this to the power of a dipole,

$$< P >= \frac{\mu_0 q_0^2 \omega^4 d^2}{12\pi c}$$

,

$$R = \frac{\mu_0 d^2}{6\pi c}\omega^2 = \frac{\mu_0 d^2}{6\pi c}\frac{4\pi^2 c}{\lambda^2} = \frac{2}{3}\pi\mu_0 c \left(\frac{d}{\lambda}\right)^2$$