PC 3231 - Electricity and Magnetism 2

AY04/05 SEM 1
Suggested Solutions

Q1

A

Drawing the Amperian loop in the question with length l,

$$\begin{aligned}
&\int \vec{B}\cdot d\vec{l} = \mu_0 \int \vec{j}\cdot da \\
\Rightarrow \quad & 2Bl = \mu_0 I \\
\Rightarrow \quad & B = \mu_0 K/2
\end{aligned}$$

, so that

$$\vec{B} = \pm\frac{\mu_0}{2}K\hat{y}$$

with the + sign for $z < 0$ and the - sign for $z > 0$.

$$\vec{B} = \nabla\times\vec{A} \Rightarrow \int \vec{B}\cdot d\vec{a} = \int \vec{A}\cdot d\vec{l}$$

Drawing a rectangular Amperian loop with its normal parallel to $\vec{B}$ and one edge of length l on the sheet where $\vec{A}$ is set to 0 and the opposite edge at distnace z away from the edge, we have

$$\begin{aligned}
&\int \vec{B}\cdot d\vec{a} = \int \vec{A}\cdot d\vec{l} \\
\Rightarrow \quad & zlB = lA \\
\Rightarrow \quad & \vec{A} = \frac{\mu_0}{2z}K\hat{x}.
\end{aligned}$$

B

For the complete ring, $\vec{B} = \mu_0\vec{M}$. For a square loop,

$$\vec{B} = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_{\cdot})$$

per segment where $s = a/2$, $\theta_2 = -\theta_1 = \pi/4$. Multiplying by 4 segments, the field of a square loop is hence

$$B_{square} = \frac{\sqrt{2}\mu_0 I}{\pi a/2}$$

where $I = WK_b = W|\vec{M} \times \hat{n}| = WM$, so

$$B_{square} = \frac{2\sqrt{2}\mu_0 MW}{\pi a}$$

. The net field in the gap is thus

$$\vec{B} = \mu_0 \vec{M} \left(1 - \frac{2\sqrt{2}W}{\pi a} \right)$$

.

Q2

A

Before the introduction of the displacement current, Ampere's law was simply

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

, implying that $\nabla \cdot (\nabla \times \vec{B}) = \mu_0 \nabla \cdot \vec{J}$. The gradient of any curl should be zero, but in general $\nabla \cdot \vec{J} \neq 0$; by the continuity equation

$$\begin{aligned} \nabla \cdot \vec{J} &= -\frac{\partial \rho}{\partial t} \\ &= -\frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \vec{E}) \\ &= -\nabla \cdot (\epsilon_0 \frac{\partial \vec{E}}{\partial t}). \end{aligned}$$

This motivates the correction to Ampere's law so that it now reads

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 (\epsilon_0 \frac{\partial}{\partial t} \vec{E})$$

so that $\nabla \cdot (\nabla \times \vec{B}) = 0$ as it should.
For the coaxial cable,

$$\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi}.$$

inside the cable and 0 outside. Drawing a rectangular Amperian loop with one edge of length l inside the coaxial cable and at a distance r away from the inner cable, and the opposite edge outside the coaxial cable, we have, by Faraday's law,

$$\oint \vec{E} \cdot d\vec{l} = El = -\frac{d\Phi}{dt}$$

, so that

$$\begin{aligned} E &= -\frac{1}{l}\frac{d}{dt}\int \vec{B} \cdot d\vec{a} \\ &= -\frac{d}{dt}\int_r^a \frac{\mu_0 I}{2\pi r'}\, dr' \end{aligned}$$

Hence

$$\vec{E}(r) = \frac{\mu_0 I_0 \omega}{2\pi} \sin \omega t \ln \frac{a}{r} \hat{z},$$

and the displacement current

$$\begin{aligned} \vec{J}_d &= \mu_0 \epsilon_0 \frac{\partial}{\partial t} \vec{E} \\ &= \frac{\epsilon_0 \mu_0 I_0 \omega^2}{2\pi} \cos \omega t \ln \frac{R}{r} \hat{z} \\ &= \frac{\mu_0 \epsilon_0}{2\pi} \omega^2 I \ln \frac{a}{r}. \end{aligned}$$

B

(i)

Consider 2 sets of potential:

$$\begin{aligned}
\vec{A}' &= \vec{A} + \vec{\alpha} \\
V' &= V + \beta
\end{aligned}$$

such that $\vec{A}$ and $\vec{A}'$ give the same $\vec{B}$ and $\vec{E}$:

$$\vec{B} = \nabla \times \vec{A} = \nabla \times \vec{A}'$$

$$\nabla \times \vec{\alpha} = 0.$$

Writing α as the gradient of a scalar λ,

$$\begin{aligned}
\alpha &= \nabla\lambda \; (\nabla \times \nabla\lambda = 0) \\
\vec{E} &= -\nabla V - \frac{\partial}{\partial t}\vec{A} \\
&= -\nabla V' - \frac{\partial}{\partial t}\vec{A}'
\end{aligned}$$

hence

$$\nabla\beta + \frac{\partial}{\partial t}\vec{\alpha} = 0$$

$$\nabla(\beta + \frac{\partial}{\partial t}\lambda) = 0$$

The term in parantheses is independent of position, but it could depend on time :

$$\beta = -\frac{\partial}{\partial t} + k(t)$$

We can absorb $k(t)$ into λ, without affecting the gradient. Hence,

$$\vec{A}' = \vec{A} + \nabla\lambda,$$

$$V' = V - \frac{\partial}{\partial t}\lambda.$$

We can add $\nabla\lambda$ to $\vec{A}$, provided we simulatenously subtract $\vec{\partial}\partial t\lambda$ from V.

(ii)

$$\begin{aligned}
\vec{E} &= -\nabla V - \frac{\partial}{\partial t}\vec{A} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \\
\vec{B} &= \nabla \times \vec{A} = 0
\end{aligned}$$

This is a set of potentials for a stationary point charge q at the origin, more usually

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0}\frac{q}{r} \\ \vec{A} &= 0. \end{aligned}$$

(iii)

Gauge transforming by λ, we have

$$\begin{aligned} V' &= V - \frac{\partial}{\partial t}\lambda = 0 - \left(-\frac{1}{4\pi\epsilon_0}\frac{q}{r}\right) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \\ \vec{A}' &= \vec{A} + \nabla\lambda = -\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{r} + \frac{-1}{4\pi\epsilon_0}qt\frac{-1}{r^2}\hat{r} = 0 \end{aligned}$$

as in 'usual' potentials of a point charge.

Q3

A

$$\begin{aligned}\vec{B} &= \frac{1}{c}\hat{z} \times \vec{E} \\ &= \frac{E_0}{c}\cos(kz - \omega t)\hat{y}.\end{aligned}$$

$$\begin{aligned}u &= \frac{1}{2} < (\epsilon_0 E^2 + \frac{1}{\mu_0}B^2) > \\ &= \frac{1}{2}(\epsilon_0 + \frac{1}{\mu_o c^2})E_0^2 \cos^2(kz - \omega t) \\ &= \epsilon_0 E_0^2 \cos^2(kz - \omega t),\end{aligned}$$

while

$$\begin{aligned}S &= \frac{1}{\mu_0}\vec{E} \times \vec{B} \\ &= \frac{1}{\mu_0 c}E_0^2 \cos^2(kz - \omega t)\hat{z} \\ &= c\epsilon_0 E_0^2 \cos^2(kz - \omega t)\hat{z} \\ &= cu\hat{z}\end{aligned}$$

$$\mathcal{P} = \frac{S}{c^2} = \frac{1}{c}\epsilon_0 E_0^2 \cos^2(kz - \omega t)\hat{z}$$

Pressure on a perfect absorber $= |\mathcal{P}|c = \epsilon_0 E_0^2 \cos^2(kz - \omega t)$.

B

(i)

$$\begin{aligned}\text{Skin depth} &= \frac{1}{k_-} \\ &= \frac{1}{\omega}\sqrt{\frac{2}{\epsilon\mu}}(\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1)^{-\frac{1}{2}} \\ &\approx \frac{1}{\omega}\sqrt{\frac{2}{\epsilon\mu}}(1 + \frac{1}{2}\left(\frac{\sigma}{\epsilon\omega}\right)^2 - 1)^{-\frac{1}{2}} \\ &= \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}\end{aligned}$$

which is independent of ω.

(ii)

$$\begin{aligned}
\text{Skin depth} &= \frac{1}{k_-} \\
&= \frac{1}{\omega}\sqrt{\frac{2}{\epsilon\mu}}(\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1)^{-\frac{1}{2}} \\
&\approx \frac{1}{\omega}\sqrt{\frac{2}{\epsilon\mu}}\sqrt{\frac{\epsilon\omega}{\sigma}} \\
&= \sqrt{\frac{2}{\omega\sigma\mu}} \\
&= \frac{1}{k} \\
&= \frac{\lambda}{2\pi}.
\end{aligned}$$

(iii)

$$\begin{aligned}
< u > &= \frac{1}{2} < (\epsilon E^2 + \frac{1}{\mu}B^2) > \\
&= \frac{1}{4}E_0^2 \mathrm{e}^{-2k_- z}(\epsilon + \frac{k^2}{\mu\omega^2}) \\
&= \frac{1}{4}E_0^2 \mathrm{e}^{-2k_- z}\epsilon\left(1+\sqrt{1+(\frac{\sigma}{\epsilon\omega})^2}\right)
\end{aligned}$$

The 1 witin the paranthesis comes from the electrical contribution and the other term, which is larger than 1, comes from the magnetic contribution. Hence, the magnetic contribution always dominates.

Q4

(i)

$$\begin{aligned}
\nabla V &= \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} \\
&= -\frac{p_0\omega}{4\pi\epsilon_0 c}\left\{\cos\theta[-\frac{1}{r^2}\sin\omega(t-r/c) - \frac{\omega}{rc}\cos\omega(t-r/c)] - \frac{\sin\theta}{r^2}\sin\omega(t-r/c)\hat{\theta}\right\} \\
&\approx \frac{p_0\omega^2}{4\pi\epsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\cos\omega(t-r/c)\hat{r},
\end{aligned}$$

$$\frac{\partial}{\partial t}\vec{A} = -\frac{\mu_0 p_0 \omega^2}{4\pi r}\cos[\omega(t-r/c)](\cos\theta\hat{r} - \sin\theta\hat{\theta}),$$

so

$$\begin{aligned}
\vec{E} &= -\nabla V - \frac{\partial \vec{A}}{\partial t} \\
&= -\frac{\mu_0 p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\omega(t-r/c)\hat{\theta}.
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
\vec{B} &= \nabla \times \vec{A} \\
&= \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{\phi} \\
&= \frac{-\mu_0 q_0\omega}{4\pi r}\left\{\frac{\omega}{c}\sin\theta\cos\omega(t-r/c) + \frac{\sin\theta}{r}\sin\omega(t-r/c)]\right\}\hat{\phi} \\
&\approx -\frac{\mu_0 p_0\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-\frac{r}{c}]\hat{\phi}.
\end{aligned}$$

(ii)

$$\begin{aligned}
<\vec{S}> &= \frac{1}{\mu_0}(<\vec{E}\times\vec{B}>) \\
&= \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right) < \cos\omega(t-r/c) >\right\}^2\hat{r} \\
&= \left(\frac{\mu_0 p_0^4\omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\hat{r}.
\end{aligned}$$

(iv)

$$
\begin{aligned}
< P > &= \int < \vec{S} > \cdot d\vec{a} \\
&= \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \int \frac{\sin^2\theta}{r^2} r^2 \sin\theta d\theta d\phi \\
&= \frac{\mu_0 p_0^2 \omega^4}{12\pi c}
\end{aligned}
$$

(v)

$$P = I^2 R = q_0^2 \omega^2 \sin^2 \omega t R$$

Average power,

$$< P >= \frac{1}{2} q_0^2 \omega^2 R$$

Equating this to the power of a dipole,

$$< P >= \frac{\mu_0 q_0^2 \omega^4 d^2}{12\pi c}$$

,

$$R = \frac{\mu_0 d^2}{6\pi c}\omega^2 = \frac{\mu_0 d^2}{6\pi c}\frac{4\pi^2 c}{\lambda^2} = \frac{2}{3}\pi\mu_0 c \left(\frac{d}{\lambda}\right)^2$$